Tweede deeltentamen Lin Alg 1 23.12.2016

1. a. $A\underline{v} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 2 & -2 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ 5 \end{bmatrix} = 5\underline{v}.$ 2

b. $\det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 11\lambda + 5. = \chi(\lambda)$

$\lambda = 5$ is eigenwaarde $\Rightarrow \chi(\lambda) = (\lambda - 5)\, g(\lambda)$ 1

Factoriseer: $(5-\lambda)(\lambda-1)^2.$ 1

Eigenruimte bij $\lambda = 1$:

$$A - I = \begin{bmatrix} 2 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Dus $\mathrm{Eig}_{\lambda=1}$ wordt opgespannen door bijv

$\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ en $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ 2

c. $\dim \mathrm{Eig}_{\lambda=1} = 2 =$ alg. mult. van $\lambda = 1$ in $\chi(\lambda)$.

idem voor $\lambda = 5$. Dus A is diag.baar. 2

Ook: $\underline{v}_1, \underline{v}_2$ en $\underline{v}_3$ vormen basis van $\mathbb{R}^3$. Dit is een alternatief antwoord. $P = [\underline{v}_1\ \underline{v}_2\ \underline{v}_3]$. $D = \mathrm{diag}(5,1,1)$

d. Nee. A is orth. diagbaar $\Leftrightarrow$ A is symmetrisch.
Hier is $A \neq A^T$ dus niet symm. 1

2. a. Laat $\underline{w}_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}$. $\|\underline{w}_1\|^2 = 36$.

$$\underline{w}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad 2 \quad \|\underline{w}_2\| = 9$$

$$\underline{w}_3 = \begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}. \quad \|\underline{w}_3\| = 9. \quad 1$$

Dus ONB is $\left\{ \frac{\underline{w}_1}{6}, \frac{\underline{w}_2}{3}, \frac{\underline{w}_3}{3} \right\} = \left\{ \begin{bmatrix} +1/3 \\ +2/3 \\ 0 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\}$!

b. Laat $Q = \frac{1}{3} \begin{bmatrix} +1 & 2 & -2 \\ +2 & 0 & 1 \\ 0 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ $\quad Q^T Q = I_3$.

Dus als $A = QR$ geldt $Q^T A = Q^T Q R = R$. 1

$$\text{Dus } R = \frac{1}{3} \begin{bmatrix} +1 & +2 & 0 & -2 \\ 2 & 0 & 2 & 1 \\ -2 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ 4 & 2 & 5 \\ 0 & 2 & 4 \\ -4 & -1 & -3 \end{bmatrix} = \begin{bmatrix} +6 & +3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

2

~~[illegible]: om $R_{11} > 0$ te krijgen nemen we als eerste kolom voor Q $\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ -2/3 \end{bmatrix}$ [illegible]~~

c. $\underline{b}_1 = \text{proj}_{\text{Col}\,A}\,\underline{b} =$ laat $Q = [\underline{q}_1\ \underline{q}_2\ \underline{q}_3]$.

$$(\underline{b}\cdot\underline{q}_1)\,\underline{q}_1 + (\underline{b}\cdot\underline{q}_2)\,\underline{q}_2 + (\underline{b}\cdot\underline{q}_3)\,\underline{q}_3$$ 1

$$= \begin{bmatrix} 0 \\ -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad \underline{b}_2 = \underline{b} - \underline{b}_1 = \begin{bmatrix} 0 \\ 4/3 \\ -2/3 \\ 4/3 \end{bmatrix}.$$! 1

3. a. Laat $A = [\underline{v}_1\ \underline{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$

Op te lossen $A^T A\,\underline{x} = A^T \underline{y}$ 1

$$\Leftrightarrow \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix} \underline{x} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

Dus $\underline{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$. 2

b. $A\underline{x} = \begin{bmatrix} +4 \\ -1 \\ 4 \end{bmatrix} \quad \underline{y} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \Rightarrow$

$$\text{dist}(\underline{y}, V) = \|\underline{y} - \text{proj}_V\,\underline{y}\|$$ 1

$$= \|\underline{y} - A\underline{x}\| = \left\|\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\| = \sqrt{2}.$$ 1

4\. a. A diag.baar $\Rightarrow \exists P, D: A = P D P^{-1}$ 1

B gelijkv. aan $A \Rightarrow \exists C : A = C^{-1} B C$ 1

Dus $B = C A C^{-1}$

$= C P D P^{-1} C^{-1}$

$= (CP)\, D\, (CP)^{-1}$ 1 B

dus B is diag. baar.

5\. De bijbehorende matrix is $A = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$ 1

(zodat $Q(\underline{x}) = \underline{x}^T A \underline{x}$).

A heeft eigenwaarden gegeven door $\chi(\lambda) = (3-\lambda)(4-\lambda) - 4 = 0$

$\lambda^2 - 7\lambda + 8 = 0$ $\quad \lambda = \frac{7}{2} \pm \frac{1}{2}\sqrt{49-32}$

$= \frac{7}{2} \pm \frac{1}{2}\sqrt{17}$. $\quad 4 < \sqrt{17} < 5$

dus $\frac{7}{2} - \frac{1}{2}\sqrt{17} > 0$. $\Rightarrow$ beide λ's strikt pos.

dus pos. def. strikt. 2

(Ook kan: $\underbrace{\det A}_{=\lambda_1 \lambda_2} = 8 > 0$, $\left.\underbrace{\operatorname{tr} A}_{=\lambda_1+\lambda_2} = 7 > 0\right\}$ $\lambda_1, \lambda_2 > 0$).

c. Nee. Als $\underline{v}_1, \underline{v}_2 \in Eig_\lambda$ geldt

$c_1\underline{v}_1 + c_2\underline{v}_2 \in Eig_\lambda$ dus hoeft zeker niet

dat $\underline{v}_1 \cdot \underline{v}_2 = 0$. Elk paar lin. onafh. vectoren

in Eig_λ spannen de eigenruimte op, en slechts

enkele staan loodrecht op elkaar. 3

(Evt concreet tegenvb: als $\underline{v}_1, \underline{v}_2$ lin. onafh

en $\underline{v}_1 \cdot \underline{v}_2 = 0$ dan $\underline{v}_1, \underline{v}_1 + \underline{v}_2$ ook lin. onafh

maar $(\underline{v}_1 \cdot (\underline{v}_1 + \underline{v}_2)) = \|\underline{v}_1\|^2 \neq 0$.)

d. $\|U\underline{x}\|^2 = (U\underline{x} \cdot U\underline{x}) = \underline{x}^T U^T U \underline{x} =$

$\underline{x}^T\underline{x} = \|\underline{x}\|^2$ 3

~~omdat~~ als U ~~een~~ orthogonale matrix

is. Maar U heeft slechts orth. kolommen!

Dus geldt niet $U^T U \overset{=}{\cancel{\neq}} I$ zomaar. Dus is het algemeen niet waar.

(1,5 pt als $U^TU = I$ aangenomen)

5 $V = C([-1,1];\mathbb{R})$. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

gegeven door $\langle \underline{u}, \underline{v} \rangle = \int_{-1}^{1} \underline{u}(x)\,\underline{v}(x)\,dx$.

a. $\langle \underline{u}, \underline{v} \rangle = \int_{-1}^{1} \underline{u}(x)\,\underline{v}(x)\,dx = \langle \underline{v}, \underline{u} \rangle$. 1

$\langle c\underline{u}, \underline{v} \rangle = \int_{-1}^{1} c\,\underline{u}\,\underline{v}\,dx = c\int \underline{u}\,\underline{v}\,dx = c\langle \underline{u}, \underline{v} \rangle$ 1

$c \in \mathbb{R}$

$$\langle \underline{u} + \underline{v}, \underline{w} \rangle = \int_{-1}^{1} (\underline{u} + \underline{v})\,\underline{w}\, dx = \int_{-1}^{1} \underline{u}\underline{w}\, dx + \int_{-1}^{1} \underline{v}\underline{w}\, dx$$

$$= \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle.$$ 1

$$\langle \underline{u}, \underline{u} \rangle = \int_{-1}^{1} \underline{u}^2(x)\, dx \geq 0 \quad \text{an}$$

since $u^2(x) \geq 0 \ \forall x$, $\int_{-1}^{1} u^2(x)\, dx = 0 \iff u^2(x) = 0 \ \forall x$. 1

(en u etnom)

b. $$\langle 1, 2x-1 \rangle = \int_{-1}^{1} 2x - 1\, dx = \left[x^2 - x \right]_{-1}^{1} = 0.$$ 1

$$\langle 1, x^2 - x + \tfrac{1}{6} \rangle = \int_{-1}^{1} x^2 - x + \tfrac{1}{6}\, dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x \Big|_{-1}^{1} = 0$$ 1

$$\langle 2x-1, x^2 - x + \tfrac{1}{6} \rangle = \int_{-1}^{1} (2x-1)(x^2 - x + \tfrac{1}{6})\, dx$$

$$= \dots = 0.$$ 1

c. $T(\underline{p}) = x\,\underline{p}'(x)$. $\qquad B = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\} = \{1, 2x-1, x^2 - x + \tfrac{1}{6}\}$.

$T(\underline{b}_1) = 0.$ ← 1,5

$T(\underline{b}_2) = 2x = \underline{b}_1 + \underline{b}_2$

$T(\underline{b}_3) = x(2x-1) = 2x^2 - x. = 2\underline{b}_3 + \tfrac{1}{2}\underline{b}_2 + \tfrac{1}{6}\underline{b}_1$

Then $$[T]_B = \begin{bmatrix} 0 & 1 & 1/6 \\ 0 & 1 & 1/2 \\ 0 & 0 & 2 \end{bmatrix}$$ 1,5