Introduction to Time Series and Dynamic Econometrics

Erratum To "Prep Exam 2019 Solutions"

October 11, 2019

The document "Prep Exam 2019 Solutions" contains an error in the solution to question 4(b). The IRF for the $\{X_t\}_{t\in\mathbb{Z}}$ is correct, however, the IRF for $\{Y_t\}_{t\in\mathbb{Z}}$ is not. The correct solution is given below.

For convenience, we define $\tilde{D}_s = (\tilde{Y}_{s-1}, \tilde{X}_s, \tilde{X}_{s-1})$. The IRF for $\{Y_t\}_{t\in\mathbb{Z}}$ at $t = s-1$, $t = s$, $t = s+1$, and $t = s+2$ is then as follows. For $t = s-1$ we obtain

$$\tilde{Y}_{s-1} = y \,.$$

(by definition of the IRF)

For $t = s$ we obtain

$$\begin{aligned}
\tilde{Y}_s &= \mathbb{E}[Y_s|\tilde{D}_s] \\
&= \mathbb{E}[\alpha + \phi Y_{s-1} + \beta_0 X_s + \beta_1 X_{s-1} + \varepsilon_s|\tilde{D}_s] \\
&\quad \text{(by definition of } Y_s\text{)} \\
&= \alpha + \phi\mathbb{E}[Y_{s-1}|\tilde{D}_s] + \beta_0\mathbb{E}[X_s|\tilde{D}_s] + \beta_1\mathbb{E}[X_{s-1}|\tilde{D}_s] + \mathbb{E}[\varepsilon_s|\tilde{D}_s] \\
&\quad \text{(by linearity of the conditional expectation)} \\
&= \alpha + \phi\tilde{Y}_{s-1} + \beta_0\tilde{X}_s + \beta_1\tilde{X}_{s-1} + \mathbb{E}[\varepsilon_s|\tilde{D}_s] \\
&\quad \text{(since } Y_{s-1}\text{ , } X_s\text{ , and } X_{s-1}\text{ are known)} \\
&= \alpha + \phi\tilde{Y}_{s-1} + \beta_0\tilde{X}_s + \beta_1\tilde{X}_{s-1} + \mathbb{E}[\varepsilon_s] \\
&\quad \text{(since } \varepsilon_s\text{ is independent of past data } Y_{s-1}\text{ and } \{X_t\}\text{ is exogenous)} \\
&= \alpha + \phi\tilde{Y}_{s-1} + \beta_0\tilde{X}_s + \beta_1\tilde{X}_{s-1} \\
&\quad \text{(since } \varepsilon_t \sim WN(0,\sigma^2)\text{)} \\
&= \alpha + \phi y + \beta_0(x+v) + \beta_1 x \\
&\quad \text{(plugging in the expressions for } \tilde{Y}_{s-1}\text{ , } \tilde{X}_s\text{ , and } \tilde{X}_{s-1}\text{)} \\
&= \alpha + \phi y + (\beta_0+\beta_1)x + \beta_0 v \,.
\end{aligned}$$

For $t = s + 1$ we obtain

$$\begin{aligned}
\tilde{Y}_{s+1} &= \mathbb{E}[Y_{s+1}|\tilde{D}_s] \\
&= \mathbb{E}[\alpha + \phi Y_s + \beta_0 X_{s+1} + \beta_1 X_s + \varepsilon_{s+1}|\tilde{D}_s] \\
&\quad \text{(by definition of } Y_{s+1}) \\
&= \alpha + \phi\mathbb{E}[Y_s|\tilde{D}_s] + \beta_0\mathbb{E}[X_{s+1}|\tilde{D}_s] + \beta_1\mathbb{E}[X_s|\tilde{D}_s] + \mathbb{E}[\varepsilon_{s+1}|\tilde{D}_s] \\
&\quad \text{(by linearity of the conditional expectation)} \\
&= \alpha + \phi\tilde{Y}_s + \beta_0\tilde{X}_{s+1} + \beta_1\tilde{X}_s + \mathbb{E}[\varepsilon_{s+1}|\tilde{D}_s] \\
&\quad \text{(by definition of } \tilde{Y}_s = \mathbb{E}[Y_s|\tilde{D}_s]\,,\ \tilde{X}_{s+1} = \mathbb{E}[X_{s+1}|\tilde{D}_s] = \mathbb{E}[X_{s+1}|\tilde{X}_s, \tilde{X}_{s-1}]\,, \text{ and } X_s \text{ is known)} \\
&= \alpha + \phi\tilde{Y}_s + \beta_0\tilde{X}_{s+1} + \beta_1\tilde{X}_s + \mathbb{E}[\varepsilon_{s+1}] \\
&\quad \text{(since } \varepsilon_{s+1} \text{ is independent of past data } Y_{s-1} \text{ and } \{X_t\} \text{ is exogenous)} \\
&= \alpha + \phi\tilde{Y}_s + \beta_0\tilde{X}_{s+1} + \beta_1\tilde{X}_s \\
&\quad \text{(since } \varepsilon_t \sim WN(0, \sigma^2)) \\
&= \alpha + \phi[\alpha + \phi y + \beta_0(x + v) + \beta_1 x] + \beta_0[\gamma_0 + (\gamma_1 + \gamma_2)x + \gamma_1 v] + \beta_1(x + v) \\
&\quad \text{(plugging in the expressions for } \tilde{Y}_s\,,\ \tilde{X}_{s+1}\,, \text{ and } \tilde{X}_s) \\
&= (1 + \phi)\alpha + \phi^2 y + [\phi(\beta_0 + \beta_1) + \beta_0(\gamma_1 + \gamma_2) + \beta_1]x + [\beta_0(\phi + \gamma_1) + \beta_1]v + \beta_0\gamma_0\,.
\end{aligned}$$

Finally, for $t = s + 2$ we obtain

$$\begin{aligned}
\tilde{Y}_{s+2} &= \mathbb{E}[Y_{s+2}|\tilde{D}_s] \\
&= \mathbb{E}[\alpha + \phi Y_{s+1} + \beta_0 X_{s+2} + \beta_1 X_{s+1} + \varepsilon_{s+2}|\tilde{D}_s] \\
&\quad \text{(by definition of } Y_{s+2}) \\
&= \alpha + \phi\mathbb{E}[Y_{s+1}|\tilde{D}_s] + \beta_0\mathbb{E}[X_{s+2}|\tilde{D}_s] + \beta_1\mathbb{E}[X_{s+1}|\tilde{D}_s] + \mathbb{E}[\varepsilon_{s+2}|\tilde{D}_s] \\
&\quad \text{(by linearity of the conditional expectation)} \\
&= \alpha + \phi\tilde{Y}_{s+1} + \beta_0\tilde{X}_{s+2} + \beta_1\tilde{X}_{s+1} + \mathbb{E}[\varepsilon_{s+2}|\tilde{D}_s] \\
&\quad \text{(by definition of } \tilde{Y}_{s+1} = \mathbb{E}[Y_{s+1}|\tilde{D}_s] \text{ and } \tilde{X}_t = \mathbb{E}[X_t|\tilde{D}_s] = \mathbb{E}[X_t|\tilde{X}_s, \tilde{X}_{s-1}]) \\
&= \alpha + \phi\tilde{Y}_{s+1} + \beta_0\tilde{X}_{s+2} + \beta_1\tilde{X}_{s+1} + \mathbb{E}[\varepsilon_{s+2}] \\
&\quad \text{(since } \varepsilon_{s+2} \text{ is independent of past data } Y_{s-1} \text{ and } \{X_t\} \text{ is exogenous)} \\
&= \alpha + \phi\tilde{Y}_{s+1} + \beta_0\tilde{X}_{s+2} + \beta_1\tilde{X}_{s+1} \\
&\quad \text{(since } \varepsilon_t \sim WN(0, \sigma^2)) \\
&= \alpha + \phi\left[(1 + \phi)\alpha + \phi^2 y + [\phi(\beta_0 + \beta_1) + \beta_0(\gamma_1 + \gamma_2) + \beta_1]x + [\beta_0(\phi + \gamma_1) + \beta_1]v + \beta_0\gamma_0\right] \\
&\qquad + \beta_0[(1 + \gamma_1)\gamma_0 + (\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2)x + (\gamma_1^2 + \gamma_2)v] + \beta_1[\gamma_0 + (\gamma_1 + \gamma_2)x + \gamma_1 v]\,. \\
&\quad \text{(plugging in the expressions for } \tilde{Y}_{s+1}\,,\ \tilde{X}_{s+2}\,, \text{ and } \tilde{X}_{s+1})
\end{aligned}$$