Midterm Exam	**X_401105: Group Theory**
VU Amsterdam	**28 March 2023 (8:30-10:30)**

Please justify your answers! Even a correct answer without full explanation scores badly.

The use of books, lecture notes, calculators, etc. is not allowed.

Question 1. Determine the element in $\mathbb{Z}/3723\mathbb{Z}$ that maps to $(\overline{2}, \overline{3})$ in $\mathbb{Z}/51\mathbb{Z} \times \mathbb{Z}/73\mathbb{Z}$ under the bijection in the Chinese remainder theorem.

Question 2. Consider the symmetric group S_{11}.

(a) Find $k \in \mathbb{N}$ such that $k \cdot 7!$ is the number of elements of order 8 in S_{11}.

(b) Let $\sigma = (1\,9\,8\,7\,3)(2\,5\,9\,3\,4\,8)(1\,3\,7)(1\,5)(5\,9\,7\,6)$. Write σ^{94} as a disjoint product of non-trivial cycles in S_{11}.

Question 3. Let G be a group. Prove that the map $\varphi : G \to G$ given by $\varphi(x) = x^{-1}$ (where x^{-1} is the inverse of x) is a group homomorphism if and only if G is abelian.

Question 4. It is given that $G = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : xz > 0,\ x,\ y,\ z \in \mathbb{R}, \right\}$ and $H = \left\{ \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} : t > 0 \right\}$ are groups under matrix multiplication.

(a) Prove the map $H \times G \to G$ given by $(h, x) \to h \cdot x := hx$ is a group action of H on G.

(b) Determine the stabiliser H_τ of the element $\tau = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ for this action.

(c) Determine the kernel of the action given in part (a).

Question 5. Consider the dihedral group $D_8 = \langle r, s \mid r^4 = s^2 = 1, sr = r^{-1}s \rangle$ and the general linear group $\mathrm{GL}_2(\mathbb{R})$ of invertible 2×2 matrices with real coefficients. Prove that the map $\varphi : D_8 \to \mathrm{GL}_2(\mathbb{R})$ given on the generators by $\varphi(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\varphi(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ extends to a group homomorphism by verifying the relations are preserved.

Question 6. Suppose that G is a group and H_i is a subgroup of G for all $i \in \mathbb{N}$.

(a) Prove that $H = \bigcap_{i \in \mathbb{N}} H_i$ is a subgroup of G.

(b) If H_1 is cyclic, prove that H is cyclic.

Maximum score per subitem

1: 10	2a: 8 2b: 8	3: 10	4a: 8 4b: 7 4c: 7	5: 10	6a: 12 6b: 10

Maximum Total = 90

Mark = 1 + (Total/10)