Midterm Exam **X_401105: Group Theory**
VU Amsterdam **29 March 2022 (15:30-17:30)**

Please justify your answers! Even a correct answer without full explanation scores badly.

The use of books, lecture notes, calculators, etc. is not allowed.

Question 1. Determine the element in $\mathbb{Z}/3127\mathbb{Z}$ that maps to $(\overline{2}, \overline{5})$ in $\mathbb{Z}/53\mathbb{Z} \times \mathbb{Z}/59\mathbb{Z}$ under the bijection in the Chinese remainder theorem.

Question 2. Consider the symmetric group S_8.

(a) Determine the number of elements of order 6 in S_8 which can be written as a disjoint product of a 3−cycle and *at least* one more nontrivial cycle (i.e. of length at least 2).

(b) Let $\sigma = (18257)(1345)(23)(687)(1876)$. Write σ^{47} as a disjoint product of nontrivial cycles in S_8.

Question 3. Prove that the dihedral group $D_8 = \langle r, s \mid r^4 = s^2 = 1, sr = r^{-1}s \rangle$ and the quaternion group $Q_8 = \langle -1, i, j, k \mid i^2 = j^2 = k^2 = -1, ijk = -1 \rangle$ are not isomorphic.

Question 4. Let $\varphi : G \to H$ be a *surjective* group homomorphism.

(a) Prove that if G is abelian then H is abelian.

(b) Prove that if $G = \langle x \rangle$ is cyclic then H is cyclic and $H = \langle \varphi(x) \rangle$.

Question 5. It is given that $G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x \in \mathbb{Q}^*, y \in \mathbb{R} \right\}$ is a group under matrix multiplication. Consider the subset B of G given by $B = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{Q} \right\}$.

(a) Determine the centraliser $C_G(B)$ of B.

(b) Determine the normaliser $N_G(B)$ of B.

Question 6. Let G be a group and $ab = ba$ for some $a, b \in G$. Consider the subset

$$H = \{x \in G \mid axb = bxa\}.$$

(a) Prove that H is closed under inverses (*Hint: Consider the inverse of the equality in* H).

(b) Prove that H is closed under products (*Hint: Write the identity element in terms of* a, b, a^{-1}, b^{-1} *using the identity* $ab = ba$).

(c) Is H a subgroup of G?

Maximum score per subitem

1: 10	2a: 8 2b: 8	3: 10	4a: 8 4b: 8	5a: 11 5b: 11	6a: 7 6b: 7 6c: 2

Maximum Total = 90

Mark = 1 + (Total/10)