Group theory - Midterm exam 2022: Solutions

(1) By the extended Euclidean algorithm, we have

$$\begin{aligned} 59-53 &= 6 & 5 &= 6-1 \\ 53-8\cdot 6 &= 5 & 53-8\cdot 6 &= 6-1 \\ 6-5 &= 1 & 53-8(59-53) &= 59-53-1, \end{aligned}$$

and so $1=-10\cdot 53+9\cdot 59$. So the class is $\overline{5\cdot(-10)\cdot 53+2\cdot 9\cdot 59}=\overline{-1588}=\overline{1539}$.

(2) (a) The order of a product of pairwise disjoint cycles is the least common multiple of the lengths of those cycles. An element in S_8 which can be written as a product of a 3−cycle and at least one more 2−cycle has order 6 if it is either a product of a 3-cycle and a 2-cycle, a product of a 3-cycle and two 2-cycles or a product of two 3-cycles and one 2-cycle (all pairwise disjoint). The number of combinations for (3-cycle)(2-cycle) is $\binom{8}{3}\frac{3!}{3}\binom{5}{2}\frac{2!}{2}=1120$, the number of combinations (3-cycle)(2-cycle)(2-cycle) is $\binom{8}{3}\frac{3!}{3}\binom{5}{2}\frac{2!}{2}\binom{3}{2}\frac{2!}{2}\frac{1}{2}=1680$, and the number of combinations (3-cycle)(3-cycle)(2-cycle) is $\binom{8}{3}\frac{3!}{3}\binom{5}{3}\frac{3!}{3}\frac{1}{2}\binom{2}{2}\frac{2!}{2}=1120$, and so in total there are 3920 such elements.

(b) $\sigma=(2\,4\,7)(3\,5\,8\,6)$ and $\sigma^{47}=(2\,4\,7)^2(3\,5\,8\,6)^3=(2\,7\,4)(3\,6\,8\,5)$.

(3) Notice that the generators r,s in D_8 have orders 4 and 2, respectively so the (two distinct) elements r^2 and s both have order 2 in D_8. However, Q_8 has only one element of order 2, namely -1.

Suppose $\varphi: D_8\to Q_8$ is an isomorphism. Then both $\varphi(r^2)$ and $\varphi(s)$ have order 2 in Q_8 and since -1 is the only element of order 2 in Q_8, we have $\varphi(r^2)=\varphi(s)$ for $r^2\neq s$. This is a contradiction since φ is injective.

Alternatively, it is possible argue that these groups have different number of elements of order 2 without explicitly arguing as in the the second para above.

(4) Suppose $\varphi: G\to H$ is a surjective homomorphism.

(a) Assume G is abelian and let $x,y\in H$. Since φ is surjective, there exists $a,b\in G$ such that $\varphi(a)=x$ and $\varphi(b)=y$.. Then

$$xy=\varphi(a)\varphi(b)\overset{\varphi\text{ homom.}}{=}\varphi(ab)\overset{G\text{ abelian}}{=}\varphi(ba)\overset{\varphi\text{ homom.}}{=}\varphi(b)\varphi(a)=yx$$

and thus H is commutative.

(b) Suppose $G=\langle x\rangle$ and consider the cyclic group $\langle\varphi(x)\rangle$ generated by $\varphi(x)$. Clearly, $\langle\varphi(x)\rangle\subseteq H$. Let $y\in H$. Since φ is surjective, there exists $g\in G$ such that $\varphi(g)=y$. Since G is cyclic $g=x^a$ for some $a\in\mathbb{Z}$. Then $y=\varphi(g)=\varphi(x^a)\overset{\varphi\text{ homom.}}{=}[\varphi(x)]^a\in\langle\varphi(x)\rangle$ and thus $H=\langle\varphi(x)\rangle$.

(5) (a) We have $\begin{bmatrix}x&y\\0&1\end{bmatrix}\in G$ is in $C_G(B)$ iff $\begin{bmatrix}x&y\\0&1\end{bmatrix}\begin{bmatrix}1&z\\0&1\end{bmatrix}=\begin{bmatrix}1&z\\0&1\end{bmatrix}\begin{bmatrix}x&y\\0&1\end{bmatrix}$ for any $\begin{bmatrix}1&z\\0&1\end{bmatrix}\in B$. This holds iff $\begin{bmatrix}x&xz+y\\0&1\end{bmatrix}=\begin{bmatrix}x&y+z\\0&1\end{bmatrix}$ iff $xz=z$ iff $x=1$. That is, $C_G(B)=\left\{\begin{bmatrix}1&y\\0&1\end{bmatrix}:y\in\mathbb{R}\right\}$.

(b) We have $A=\begin{bmatrix}x&y\\0&1\end{bmatrix}\in G$ is in $N_G(B)$ iff $B=\left\{\begin{bmatrix}x&y\\0&1\end{bmatrix}\begin{bmatrix}1&z\\0&1\end{bmatrix}\begin{bmatrix}x&y\\0&1\end{bmatrix}^{-1}:z\in\mathbb{Q}\right\}$.

Then $\begin{bmatrix}x&y\\0&1\end{bmatrix}\begin{bmatrix}1&z\\0&1\end{bmatrix}\begin{bmatrix}x&y\\0&1\end{bmatrix}^{-1}=\begin{bmatrix}x&xz+y\\0&1\end{bmatrix}\begin{bmatrix}1/x&-y/x\\0&1\end{bmatrix}=\begin{bmatrix}1&xz\\0&1\end{bmatrix}\in B$ since

$xz \in \mathbb{Q}$. Also any $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \in B$, we have $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & y \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ and thus $N_G(B) = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x \in \mathbb{Q}^*,\ y \in \mathbb{R} \right\} = G$.

(6) (a) Suppose $x \in H$. Then $axb = bxa$. Taking inverses of both sides,

$$\begin{aligned} (axb)^{-1} = (bxa)^{-1} &\Leftrightarrow b^{-1}x^{-1}a^{-1} = a^{-1}x^{-1}b^{-1} \\ &\Leftrightarrow abb^{-1}x^{-1}a^{-1}ab = aba^{-1}x^{-1}b^{-1}ab \\ &\Leftrightarrow a(bb^{-1})x^{-1}(a^{-1}a)b = baa^{-1}x^{-1}b^{-1}ba \\ &\Leftrightarrow ax^{-1}b = b(aa^{-1})x^{-1}(b^{-1}b)a = bx^{-1}a \end{aligned}$$

and thus $x^{-1} \in H$.

(b) Suppose $x, y \in H$. Then $axb = bxa$ and $ayb = bya$ by the definition of H. Now consider

$$\begin{aligned} a(xy)b &= ax(bb^{-1})(a^{-1}a)yb \\ &= (axb)b^{-1}a^{-1}(ayb) \\ &= (bxa)b^{-1}a^{-1}(bya) \\ &= bx(ab^{-1}a^{-1}b)ya \\ &= b(xy)a \end{aligned}$$

since $ab = ba$ implies that $a^{-1}bab^{-1} = 1$.

(c) Yes, since $1 \in H$ and it is closed under inverses and products by parts (a) and (b).