Faculty of Science	Group theory (X_401105), part 1
Vrije Universiteit Amsterdam	Partial examination 27-3-2018 (12:00-14:00)

- **Attempt all problems.**
- Answers without reasoning score poorly, so give proper justifications everywhere.
- In case you cannot do a part of a problem, you may still use its stated result in the remainder of the problem.
- Calculators, notes, books, etc., may not be used.

(1) Compute which element $\overline{0}, \overline{1}, \ldots, \overline{3110}$ in $\mathbb{Z}/3111\mathbb{Z}$ under the map of the Chinese remainder theorem is mapped to $(\overline{3}, \overline{10})$ in $\mathbb{Z}/51\mathbb{Z} \times \mathbb{Z}/61\mathbb{Z}$.

(2) (a) Determine the number of elements in S_7 of order 4.
(b) Write $\sigma = (1\,6\,5\,2\,4\,3)(1\,2\,7\,5)$ in S_8 as a product of pairwise disjoint cycles of length at least 2.

(3) Let $G = D_{16} = \{e, r, r^2, \ldots, r^7, s, sr, sr^2, \ldots, sr^7\}$ be the dihedral group with 16 elements. In G, we consider the subset $H = \{r^{2i} \text{ with } i \text{ in } \mathbb{Z}\} \cup \{sr^{2j+1} \text{ with } j \text{ in } \mathbb{Z}\}$.
(a) Show that H is a subgroup of G. *Hint: Perform the required calculations with types of elements r^l and sr^l.*
(b) Is H Abelian?

(4) Let $\varphi : G \to H$ be an isomorphism of groups G and H.
(a) Show that, for each g in G, the orders of g and $\varphi(g)$ are the same.
(b) Prove that the inverse map $\varphi^{-1} : H \to G$ is a homomorphism.

(5) **Parts (a), (b) and (c) in this problem are independent of each other.**
Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ with } a \text{ in } \mathbb{Q}^* \text{ and } b \text{ in } \mathbb{Q} \right\}.$$

It is given that G is a subgroup of $GL_2(\mathbb{Q})$, the group of invertible 2×2-matrices with coefficients in $\mathbb{Q}$.
(a) (i) Prove, by means of induction with respect to n, that for $n \geq 2$ we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} a^n & (a^{n-1} + a^{n-2} + \cdots + a + 1)b \\ 0 & 1 \end{pmatrix}.$$

(ii) Determine the order of each element of G of finite order.

In G we define the subset

$$A = \left\{ \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} \text{ in } G \text{ with } b \text{ in } \mathbb{Z} \right\}.$$

(b) Compute $C_G(A)$.
(c) Compute $N_G(A)$.

(6) Determine the last two digits of 27^{2018}. *Hint: This can be done efficiently by, for example, a (clever) calculation in $\mathbb{Z}/100\mathbb{Z}$.*

Distribution of points					
1: 8	2a: 8 2b: 6	3a: 8 3b: 4	4a: 10 4b: 8	5ai: 6 5aii: 6 5b: 8 5c: 10	6: 8
Maximum total = 90					
Exam grade = 1 + Total/10					