Vrije Universiteit Amsterdam **22 December 2022 18:30-21:30**
Final Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**10+8+7+8 Points**] Let $f \in \mathcal{M}(\mathbb{R})$ be the function given by

$$f(x) = \begin{cases} x^2 e^{-2\pi x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

a) Show that the Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ is given by

$$\widehat{f}(\xi) = \frac{1}{4\pi^3} \frac{1}{(1+i\xi)^3}.$$

b) Give the definition of $\mathcal{M}(\mathbb{R})$ and show that $\widehat{f} \in \mathcal{M}(\mathbb{R})$.

c) Compute

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i \xi}}{(1+i\xi)^3} d\xi.$$

d) Use Plancherel's formula to show that

$$\int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^3} d\xi = \frac{3\pi}{8}.$$

Question 2. [**3+5+7+5 Points**] Let $f \in \mathcal{S}(\mathbb{R})$. In this question you are asked to prove Theorem 5.1.3 of the book.

a) Give the definition of the Schwartz space $\mathcal{S}(\mathbb{R})$.

b) Show that the Fourier transform of f is bounded. That is, show that there exists a constant $C > 0$ such that $|\widehat{f}(\xi)| \leq C$ for all $\xi \in \mathbb{R}$.

c) Compute the Fourier transform of the function $g \in \mathcal{S}(\mathbb{R})$ defined by

$$g(x) = \frac{1}{(2\pi i)^k} \frac{d^k}{dx^k} \left((-2\pi i x)^l f(x) \right)$$

in terms of the Fourier transform of f. *You may use the transformation rules*

d) Show that $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

Question 3. [7+10+10+5+5 Points] Consider the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \text{with initial conditions} \\ u(x,0) = f(x), \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x). \end{cases} \tag{1}$$

for some $f, g \in \mathcal{S}(\mathbb{R})$. We will assume that for every $t \in \mathbb{R}$ the function $x \mapsto u(x,t) \in \mathcal{S}(\mathbb{R})$, hence that the Fourier transform

$$\widehat{u}(\xi, t) := \int_{-\infty}^{\infty} u(x,t) e^{-2\pi i x \xi} dx$$

is well-defined.

a) Show formally that $\widehat{u}$ satisfies the ordinary differential equation

$$\frac{\partial^2 \widehat{u}}{\partial t^2}(\xi, t) = -4\pi^2 \xi^2 \widehat{u}(\xi, t). \tag{2}$$

b) Show that a solution to (2), for $\xi \neq 0$ is given by

$$\widehat{u}(\xi, t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t)$$

and compute $A(\xi)$ and $B(\xi)$ in terms of the Fourier transforms of f and g.

Hence $u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(\xi,t) e^{2\pi i \xi x} d\xi$ is a solution to the wave equation. In the remaining part of the question we are going to investigate the uniqueness question. Suppose that u solves the Equation (1). Define the energy E of u by

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial t}(x,t) \right|^2 + \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx$$

c) Show that $\frac{dE}{dt}(t) = 0$ hence that E is constant. *You may swap integrals and derivatives without justification*

d) Use part c) to show that if u solves (1) and $f(x) = g(x) = 0$ for all $x \in \mathbb{R}$, that then $u(x,t) = 0$ for all $x, t \in \mathbb{R}$.

e) Show that if u, v both solve (1) with the same initial conditions $u(x,0) = v(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = \frac{\partial v}{\partial t}(x,0) = g(x)$, that then $u(x,t) = v(x,t)$ for all $x, t \in \mathbb{R}$.

Question 4. [This question is a bonus question about the first part of the course. It can award you .5+.5=1 total bonus point on your grade for the midterm.] Recall that $\mathcal{R} := \mathrm{RI}([0,1], \mathbb{C})$, the space of Riemann integrable functions on $[0,1]$, has an inner product $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} dx$. Let $\{e_k\}_{k \in \mathbb{N}}$ with $e_k \in \mathcal{R}$ be orthonormal functions. Let $f \in \mathcal{R}$ and $a_k := \langle f, e_k \rangle$. Define $S_N(f) = \sum_{k=1}^N a_k e_k$.

a) Show that $\sum_{k=1}^N |a_k|^2 = \|f\|^2 - \|f - S_N(f)\|^2$.

b) Show that $a_k \to 0$ as $k \to \infty$.