Vrije Universiteit Amsterdam | **25 October 2022 18:45-21:30**
Midterm Fourier Analysis | **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**3+4+8+5+5+5 points**] Consider the 2π-periodic function f which is given for $-\pi \leq x < \pi$ by $f(x) = \cos(ax)$.

a) Sketch the graph of f on $[-3\pi, 3\pi]$ for $a = 1/2$.

For the rest of the exercise let $a \in \mathbb{R} \setminus \mathbb{Z}$.

b) Show that f is not differentiable on the circle.

c) Show that the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ of f are given by

$$\widehat{f}(n) = \frac{(-1)^n}{\pi}\frac{a\sin(a\pi)}{a^2 - n^2}.$$

d) Show that the Fourier series of f converges absolutely, hence uniformly.

e) Show that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} = \frac{\pi}{a\sin(a\pi)},$$

and compute the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2 - n^2}.$$

e) Use Parseval's identity to compute

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a^2 - n^2)^2}.$$

Question 2. [**3+10 points**] Let $f : \mathbb{R} \to \mathbb{C}$ be a differentiable function with continuous derivative which is 2π periodic. Let $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ be the Fourier coefficients of f.

a) Let $g : \mathbb{N} \to \mathbb{C}$ be a function. Give the definition of $\widehat{f}(n) = O(g(n))$ as $n \to \infty$.

b) Show that $\widehat{f}(n) = O(1/n)$ as $n \to \infty$.

Continued on the back ⇒ ...

Question 3. [**15 points**] Suppose $f : \mathbb{R} \to \mathbb{C}$ is a Riemann integrable function such that $f(x) = f(x + \frac{2\pi}{3})$ for all x. Show that the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ can be written as

$$\widehat{f}(n) = \frac{a(n)}{2\pi}\int_{-\pi/3}^{\pi/3} f(x)e^{-inx}dx,$$

for coefficients $a(n)$ which do not depend on f. Compute $a(n)$ and show that $a(n) = 0$, whenever $n \neq 0 \pmod 3$.

Question 4. [**10 points**] Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic Riemann integrable function. Suppose the Fourier series of f converges pointwise to f. Suppose that f is *not* continuous. Give a short argument that the Fourier series does not converge uniformly to f.

Question 5. [**5+10+7 points**] Let $w : [0, 1] \to \mathbb{R}$ a continuous function with $w(x) > 0$ for all $x \in [0, 1]$. Let $\mathcal{R} = \mathrm{RI}([0, 1], \mathbb{C})$ be the space of complex valued Riemann integrable functions on $[0, 1]$. Define the Hermitian inner product on $\mathcal{R}$ by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}w(x)dx,$$

for $f, g \in \mathcal{R}$. The associated norm is denoted by $\|f\| = \sqrt{\langle f, f\rangle}$.

a) Show that the inner product is not strictly positive definite: Give an example of a function $f \in \mathcal{R}$ such that $\|f\| = 0$, but f is not the zero function.

b) Show that the inner product *is* strictly positive definite if we restrict it to $C^0([0, 1], \mathbb{C}) \subset \mathcal{R}$, the space space of continuous functions. That is, show that if $\|f\| = 0$ for some $f \in C^0([0, 1], \mathbb{C})$, that then $f(x) = 0$ for all $x \in [0, 1]$.

Let $w(x) = e^x$. Consider the constant function $P_0(x) = 1$ and the degree one polynomial $P_1(x) = 1 + ax$.

c) Compute the number a such that P_0 and P_1 are orthogonal.