Vrije Universiteit Amsterdam — **14 Februari 2022 18:45-21:30**

Final Fourier Analysis Resit full course — **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

THIS IS THE RESIT FOR THE FULL COURSE. IF YOU EMAILED ME THAT YOU ONLY WANT TO RESIT THE MIDTERM YOU SHOULD NOT MAKE THIS EXAM

Question 1. 8+5+5+8 points Consider the function $f \in \mathcal{M}(\mathbb{R})$ given by

$$f(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

a) Show that the Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ is given by

$$\widehat{f}(\xi) = \frac{1}{(1+2\pi i\xi)^2}.$$

b) Give the definition of $\mathcal{M}(\mathbb{R})$ and show that $\widehat{f} \in \mathcal{M}(\mathbb{R})$

c) Show that

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i\xi}}{(1+2\pi i\xi)^2} d\xi = \frac{1}{e}.$$

d) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1+2\pi i n)^2} = \frac{e}{(e-1)^2}$$

Hint: You may use that $\sum_{n=0}^{\infty} ny^n = y\frac{d}{dy}\sum_{n=0}^{\infty} y^n$ *whenever* $|y| < 1$

Question 2. 2+2+5+5+5 Points Let $f \in \mathcal{S}(\mathbb{R})$.

a) Give the definition of $\mathcal{S}(\mathbb{R})$.

b) Let $g : \mathbb{R} \to \mathbb{C}$ satisfy $\lim_{x\to-\infty} g(x) = c$ with $c \neq 0$. Show that $g \notin \mathcal{S}(\mathbb{R})$.

Let $g_1(x) = \int_x^{\infty} f(y)dy$ and $g_2(x) = \int_{-\infty}^{x} f(y)dy$. By part b) we have that if $\int_{-\infty}^{\infty} f(y)dy \neq 0$ that $g_1, g_2 \notin \mathcal{S}(\mathbb{R})$. From now on we assume that $\int_{-\infty}^{\infty} f(y)dy = 0$.

c) Show that $\lim_{x\to\infty} |x^k g_1(x)| = 0$ for all $k \geq 0$.
Hint: Note that $|x^k \int_x^{\infty} f(y)dy| \leq \int_x^{\infty} |y^k f(y)|dy$ *when* $x \geq 0$.

d) Show that $g_1, g_2 \in \mathcal{S}(\mathbb{R})$.

e) Show that for $\xi \neq 0$ that

$$\mathcal{F}(g_1)(\xi) = \frac{-1}{2\pi i\xi}\mathcal{F}(f)(\xi).$$

Exam continued on the back $\Rightarrow \ldots$

Question 3. 3+9 Points Let $k \in \mathbb{N}$ and let $f_k : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{C}$ be functions.

a) Give the (ϵ, N)-definition of the statement: "The sequence f_k converges uniformly to f as $k \to \infty$".

b) Suppose that all f_k are continuous and that f_k converges uniformly to f. Show that f is continuous.

Question 4. 5+9+9+10 points Let f be a 2π-periodic Riemann integrable function.

a) Show that there exist a constant C such that the Fourier coefficients

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

satisfy $|\widehat{f}(n)| \leq C$ for all n.

By modeling the temperature u of a metal ring, one is led to the differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

with initial condition $u(x,0) = f(x)$, where f is as above. We will assume that u is 2π-periodic in the x variable.

b) Suppose that u has a Fourier expansion in the x variable of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \widehat{u}(n,t) e^{inx}.$$

Show that (formally) the fourier coefficients $\widehat{u}(n,t)$ obey the differential equation

$$\frac{\partial \widehat{u}}{\partial t}(n,t) = -n^2 \widehat{u}(n,t)$$

with initial condition $\widehat{u}(n,0) = \widehat{f}(n)$. Conclude that the solution to this equation is $\widehat{u}(n,t) = \widehat{f}(n) e^{-n^2 t}$.

c) Show that

$$u(x,t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{-n^2 t} e^{inx}$$

converges uniformly in x for every $t > 0$. *Hint: use part a).*

d) Show that

$$\lim_{t \searrow 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x,t) - f(x)|^2 dx = 0,$$

where u is as in part c). *Hint: use Parseval's identity and show that the sum so obtained converges uniformly in t.*