Vrije Universiteit Amsterdam **16 December 2021 18:45-21:30**
Final Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. 2+9+6+6+6+6 Points Define the function

$$f(x) = \begin{cases} \frac{\pi}{2} - 2\pi^3 x^2 & -\frac{1}{2\pi} \leq x \leq \frac{1}{2\pi} \\ 0 & \text{otherwise.} \end{cases}$$

a) Sketch the function f.

b) Show that the Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ is

$$\widehat{f}(\xi) = \begin{cases} \frac{\sin(\xi) - \xi\cos(\xi)}{\xi^3} & \xi \neq 0 \\ \frac{1}{3} & \xi = 0. \end{cases}$$

Hint: You may use transformation rules without justification.

This function is continuous, which you do not have to show.

c) Give the definition of $\mathcal{M}(\mathbb{R})$ and show that $\widehat{f} \in \mathcal{M}(\mathbb{R})$.

d) Show that

$$\int_0^{\infty} \frac{\sin(\xi)\cos(\xi) - \xi\cos^2(\xi)}{\xi^3} d\xi = 0.$$

e) Compute

$$\int_{-\infty}^{\infty} \frac{(\sin(\xi) - \xi\cos(\xi))^2}{\xi^6} d\xi.$$

f) Show that

$$\sum_{n=1}^{\infty} \frac{\sin(n) - n\cos(n)}{n^3} = \frac{\pi}{4} - \frac{1}{6}.$$

Question 2. 8+7 Points A probability distribution on $\mathbb{R}$ is a non-negative function $f \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $\delta > 0$ and let f be a probability distribution and define $K_\delta(x) = \frac{1}{\delta} f\left(\frac{x}{\delta}\right)$. You are going to prove that K_δ is a family of good kernels on $\mathbb{R}$ as $\delta \to 0$.

a) Show that $\int_{-\infty}^{\infty} K_\delta(x)dx = 1$ for all $\delta > 0$.

b) Show that for every $\eta > 0$, we have that

$$\int_{|x|>\eta} K_\delta(x)dx \to 0 \qquad \text{as} \qquad \delta \to 0.$$

Question 3. 10+10+5 Points Let $f, w \in \mathcal{S}(\mathbb{R})$. Consider the integro-differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \int_{-\infty}^{\infty} w(x-y)u(y,t)dy \\ u(x,0) &= f(x). \end{cases} \tag{1}$$

This is a model for the concentration of nutrients in a soil, which are absorbed by roots of plants. In this exercise you may assume that if u solves (1), that then $u(\cdot,t) \in \mathcal{S}(\mathbb{R})$ for all $t \geq 0$.

a) Let $\widehat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{-2\pi i x\xi}dx$ be the Fourier transform of u in the x variable. Show that $\widehat{u}$ then satisfies the equation

$$\begin{cases} \frac{\partial \widehat{u}}{\partial t}(\xi,t) &= -4\pi^2\xi^2\widehat{u}(\xi,t) + \widehat{w}(\xi)\widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) &= \widehat{f}(\xi). \end{cases} \tag{2}$$

b) Compute the solution $\widehat{u}$ to Equation (2).

c) Use part b) to write down the solution u to Equation (1).

Question 4. 8+7 Points Let $f(x) = e^{-\pi x^2}$. In this exercise you may use the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$ and that $f \in \mathcal{S}(\mathbb{R})$.

a) Show that the Fourier transform $g(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}dx$ obeys the differential equation

$$g'(\xi) = -2\pi\xi g(\xi).$$

b) Show that $g(\xi) = e^{-\pi\xi^2}$.