Vrije Universiteit Amsterdam **26 October 2021 18:45-21:30**
Midterm Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**3+10+6+4+7 Points**] Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ with $\alpha > 1$. Consider the 2π-periodic function f which is given for $x \in [-\pi, \pi]$ by

$$f(x) = \begin{cases} \sin(\alpha x) & -\pi/\alpha \leq x \leq \pi/\alpha \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graph of the function f on the interval $[-3\pi, 3\pi]$ for $\alpha = \frac{3}{2}$.

b) Show that the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ are given by

$$\widehat{f}(n) = \frac{i\alpha \sin\left(\frac{n\pi}{\alpha}\right)}{\pi(n^2 - \alpha^2)}.$$

c) Argue that the Fourier series of f converges everywhere to f. You may state theorems that you use, however you must state the correct hypothesis of these theorems.

d) Show that

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{\alpha}\right)}{(n^2 - \alpha^2)} \sin(nx) = 0,$$

for all $\pi/\alpha \leq x \leq \pi$.

e) Compute the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(\frac{n\pi}{\alpha})}{(n^2 - \alpha^2)^2}.$$

Question 2. [**8+8 Points**] Let f be a 2π-periodic Riemann integrable function. Let $a \in \mathbb{R}$ and $k \in \mathbb{Z}$. Define the functions $g(x) = f(x - a)$ and $h(x) = e^{ikx} f(x)$.

a) Show that $\widehat{g}(n) = e^{-ina} \widehat{f}(n)$.

b) Compute the Fourier coefficients of h in terms of the Fourier coefficients of f.

Question 3. [**12 Points**] True or false. You do not need to explain your answers. You will get points if you have ≥ 3 answers correct. For each correct answer ≥ 3 you will get 3 points.

a) The Fourier series of a 2π-periodic continuous function converges pointwise at every point.

b) If the Fourier series of a 2π-periodic continuous function f converges at a point x, it converges to $f(x)$.

c) A summable series is also Cesáro summable.

d) The Dirichlet kernel $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ is a good kernel.

e) The Fejer kernel $F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_N(x)$ is a good kernel.

f) A differentiable function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Question 4. [**5+10 Points**] Let $(V, \langle\cdot,\cdot\rangle)$ be a hermitean inner product space. Let $e_n \in V$ for $n \in \mathbb{N}$ be orthonormal vectors. This means that $\langle e_n, e_m\rangle = \delta_{nm}$ where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}.$$

Let $X \in V$. Define $P_N(X) = \sum_{n=1}^{N} \langle X, e_n\rangle e_n$.

a) Show that $X - P_N(X) \perp e_k$ for all $1 \leq k \leq N$.

b) Prove the best approximation lemma in this context: i.e. show that

$$\|X - P_N(X)\| \leq \|X - \sum_{n=1}^{N} c_n e_n\|,$$

for all $c_n \in \mathbb{C}$, with equality if and only if $c_n = \langle X, x_n\rangle$.

Question 5. [**17 Points**] Let,

$$G_N(x) = \sum_{n=-N}^{N} \mathrm{sign}(n) e^{inx} \qquad \text{where} \qquad \mathrm{sign}(n) = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \\ -1 & n < 0 \end{cases}.$$

Show that

$$G_N(x) = i\frac{\cos(x/2) - \cos((N + 1/2)x)}{\sin(x/2)},$$

for $x \neq 0$.