Vrije Universiteit Amsterdam — **12 april 2021 18:45-21:30**

Second resit Fourier Analysis — **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**8+7+7+8 Points**] Define the function

$$f(x) = \begin{cases} \sin\left(\frac{x}{2}\right) & -2\pi \leq x \leq 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

It is true that $f \in \mathcal{M}(\mathbb{R})$ which you do not have to prove.

a) Show that the Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ is given by

$$\widehat{f}(\xi) = \begin{cases} \frac{4i\sin(4\pi^2\xi)}{16\pi^2\xi^2-1} & \xi \neq \pm\frac{1}{4\pi} \\ \mp 2i\pi & \xi = \pm\frac{1}{4\pi}. \end{cases}$$

You may use that $\widehat{f}$ is continuous at $\xi = \pm\frac{1}{4\pi}$ without proof below.

b) Give the definition of $\mathcal{M}(\mathbb{R})$ and show that $\widehat{f} \in \mathcal{M}(\mathbb{R})$.

c) Show that

$$\int_0^{\infty} \frac{\sin(4\pi^2\xi)\sin(2\pi^2\xi)}{1-16\pi^2\xi^2}d\xi = \frac{1}{8}.$$

d) Use Plancherel's formula to compute

$$\int_{-\infty}^{\infty} \frac{\sin^2(4\pi^2\xi)}{(1-16\pi^2\xi^2)^2}d\xi.$$

Question 2. [**7+8 Points**]Let $k \in \mathbb{N}$. For this exercise it is good to recall Newton's Binomial formula $(a+b)^k = \sum_{n=0}^{k}\binom{k}{n}a^n b^{k-n}$, where $\binom{k}{n} = \frac{k!}{n!(k-n)!}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (1+e^{ix})^k$.

a) Compute the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ of the function f.

b) Use Parseval's identity to prove that the combinatorial identity

$$\sum_{n=0}^{k}\binom{k}{n}^2 = \binom{2k}{k}$$

holds. *Hint: Note that* $|1+e^{ix}|^{2k} = \left(e^{\frac{ix}{2}} + e^{-\frac{ix}{2}}\right)^{2k}$.

Question 3. [**7+8 Points**] Let $f : [-\pi, \pi] \to \mathbb{C}$ be an odd continuous function.

a) Show that the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ obey $\widehat{f}(-n) = -\widehat{f}(n)$

b) Suppose that the Fourier series converges uniformly. Show that we can write

$$f(x) = \sum_{n=1}^{\infty} a(n)\sin(nx)$$

for coefficients $a(n)$. Express $a(n)$ in terms of f.

Question 4. [**8+6+7+9 Points**] Suppose that there is a unique smooth function $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ that satisfies the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = -\frac{\partial^4 u}{\partial x^4}(x,t) \quad \text{for} \quad t > 0,$$

subject to the initial condition $u(x, 0) = f(x)$ for some given function f. We will assume that for every $t \geq 0$ that $u(\cdot, t) \in \mathcal{S}(\mathbb{R})$. In particular $f \in \mathcal{S}(\mathbb{R})$. Let $\widehat{u}$ denote the Fourier transform of u in the x variable.

a) Show that $\widehat{u}$ satisfies

$$\frac{\partial \widehat{u}}{\partial t}(\xi, t) = -(2\pi)^4\xi^4\widehat{u}(\xi, t)$$

and $\widehat{u}(\xi, 0) = \widehat{f}(\xi)$. Compute the solution $\widehat{u}$.

b) Let $c > 0$. Show that the function g_c given by $g_c(\xi) = e^{-c\xi^4}$ satisfies

$$\sup_{\xi\in\mathbb{R}} |\xi^k g_c(\xi)| < \infty$$

for all $k \in \mathbb{N}$.

c) Give the definition of $\mathcal{S}(\mathbb{R})$ and show that $g_c \in \mathcal{S}(\mathbb{R})$. *Hint: Show that for every l there exists a polynomial P_l such that $g_c^{(l)}(\xi) = P_l(\xi)e^{-c\xi^4}$ and use part b).*

d) Show that there is a function $K \in \mathcal{S}(\mathbb{R})$ such that

$$u(x,t) = \frac{1}{t^{1/4}}\int_{-\infty}^{\infty} f(y)K\left(\frac{x-y}{t^{1/4}}\right)dy$$

for all $t > 0$.