Vrije Universiteit Amsterdam **8 February 2021 18:45-20:45**
Resit Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**5+10+5+8+7 Points**] Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function, which for $x \in (-\pi, \pi]$ is given by

$$f(x) = \begin{cases} \cos(x) & 0 < x < \pi \\ 0 & x = 0, \pi \\ -\cos(x) & -\pi < x < 0 \end{cases}$$

a) Sketch the graph of f. Is f odd, or even or neither?

b) Show that the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ of f are given by

$$\widehat{f}(n) = \begin{cases} \frac{2}{i\pi} \frac{n}{n^2-1} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

c) Argue that the Fourier series of f converges everywhere to f. You may state theorems that you use, however you must state the correct hypotheses.

d) Show that

$$\cos(x) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k}{4k^2 - 1} \sin(2kx).$$

for $0 < x < \pi$.

e) Compute

$$\sum_{k=1}^{\infty} \frac{k^2}{(4k^2 - 1)^2}.$$

Question 2. [**7+8 Points**] Let f be an integrable function on the circle.

a) Give a proof of the Riemann-Lebesgue lemma, that is, show that $\lim_{n \to \pm\infty} |\widehat{f}(n)| = 0$.

Let g be an integrable function on the circle and suppose that there exists an integrable function on the circle f such that

$$f * g = g.$$

b) Show that there exists an $N > 0$ such that $\widehat{g}(n) = 0$ for all n with $|n| \geq N$, i.e. show that g is a trigonometric polynomial.

Question 3. [**10+6+8+5+6+5 Points**] Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \tag{1}$$

on the half space $\{(x,y)\,|\,y \geq 0\}$ subject to the boundary condition $u(x,0) = f(x)$ for some $f \in \mathcal{S}(\mathbb{R})$. If we consider the Fourier transform of u in the x variable, we guess the solution $\widehat{u}(\xi, y) = \widehat{f}(\xi)h_y(\xi)$ by the standard arguments, where the function h_y is given by $h_y(\xi) = e^{-2\pi|\xi|y}$. It is true that $h_y \in \mathcal{M}(\mathbb{R})$ for $y > 0$ which you do not need to prove.

a) Show that the **inverse** Fourier transform of h_y for $y > 0$ is the Poisson kernel $\mathcal{P}_y$ on the real line, which is given by

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

b) State the definition of $\mathcal{M}(\mathbb{R})$ and show that $\mathcal{P}_y \in \mathcal{M}(\mathbb{R})$. Conclude that the Fourier transform of $\mathcal{P}_y$ equals h_y.

We therefore guess that the solution to (1) is given by

$$u(x,y) = \mathcal{P}_y * f(x) \quad \text{for} \quad y > 0,$$

and that this function satisfies the boundary condition in the sense that $\lim_{y\to 0} u(x,y) = f(x)$. You are going to show this below.

c) Show explicitly that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} f(t)dt,$$

solves (1) for $y > 0$. You are allowed to swap integrals and derivatives without justification.

d) Show that $\int_{-\infty}^{\infty} \mathcal{P}_y(x)dx = 1$ for $y > 0$.

Hint: It is easy if you use the Fourier transform.

e) Show that for every $\eta > 0$ we have that

$$\int_{|x|>\eta} \mathcal{P}_y(x)dx \to 0 \qquad \text{as} \qquad y \to 0.$$

f) Explain why $\lim_{y\to 0} \mathcal{P}_y * f(x) = f(x)$.