Vrije Universiteit Amsterdam — **18 December 2020 12:15-14:15**
Final Fourier Analysis — **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10}+1$. *Recall that the Fouriertransform is defined by the formula* $\widehat{f}(\xi)=\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$.

Question 1. [**7+7+7+7+7 points**] Define the function

$$f(x)=\begin{cases}\frac{\pi}{2}\cos(\pi x) & -\frac{1}{2}\leq x\leq \frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$

The function $f\in\mathcal{M}(\mathbb{R})$ which you do not have to prove.

a) Show that

$$\hat{f}(\xi)=\begin{cases}\frac{\pi}{4} & \xi=\pm\frac{1}{2}\\ \frac{\cos(\pi\xi)}{1-4\xi^2} & \text{otherwise.}\end{cases}$$

The function $\hat{f}$ is continuous, which you do not have to prove.

b) Give the definition of $\mathcal{M}(\mathbb{R})$ and show that $\hat{f}\in\mathcal{M}(\mathbb{R})$.

c) Show that

$$\int_0^\infty \frac{\cos(\pi\xi)}{1-4\xi^2}d\xi=\frac{\pi}{4}.$$

d) Show that

$$\int_{-\infty}^\infty \frac{\cos^2(\pi\xi)}{(1-4\xi^2)^2}d\xi=\frac{\pi^2}{8}.$$

e) Use Poisson's summation formula to compute

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-4n^2}.$$

Question 2. [**7+8 Points**] Let $f_1(x)=e^{-\pi x^2}$. Recall that we have proven that $\widehat{f_1}(\xi)=e^{-\pi\xi^2}$ in the lectures, which you can use in this exercise. Define $f_{n+1}=f_1 * f_n$ for all $n\in\mathbb{N}$.

a) Show that $\widehat{f_n}(\xi)=e^{-n\pi\xi^2}$.

b) Show that $f_n(x)=\frac{1}{\sqrt{n}}e^{-\frac{\pi x^2}{n}}$.

Question 3. [**8+7+5+5+7+8 Points**] Consider the the following differential equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0. \tag{1}$$

We will take (x, y) to lie in the upper half plane $\{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$. We will impose the boundary condition

$$u(x, 0) = f(x) \tag{2}$$

for some function $f \in \mathcal{S}(\mathbb{R})$.

a) Suppose u satisfies (1). Take (formally) the Fourier transform of u in the x variable. Show that this Fourier transform satisfies

$$-4\pi^2\xi^2\widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) = 0. \tag{3}$$

b) Show that the solutions of (3) are of the form

$$\widehat{u}(\xi, y) = A(\xi)e^{-2\pi|\xi|y} + B(\xi)e^{2\pi|\xi|y},$$

for unknown functions $A(\xi)$ and $B(\xi)$.

From now on we will assume that $B(\xi) = 0$.

c) Argue that if u satisfies the boundary condition (2) that then $A(\xi) = \widehat{f}(\xi)$.

d) Check that

$$u(x, y) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{-2\pi|\xi|y}e^{2\pi ix\xi}d\xi, \tag{4}$$

is a solution to (1). You are allowed to swap integrals and derivatives without justification.

Plancherel's formula applies to the function $u(\cdot, y)$ for every y in (4), which you do not have to prove.

e) Use Plancherel's formula to show that

$$\int_{-\infty}^{\infty} |u(x, y) - f(x)|^2dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2|1 - e^{-2\pi|\xi|y}|^2d\xi. \tag{5}$$

f) Prove carefully that $\int_{-\infty}^{\infty} |u(x, y) - f(x)|^2dx \to 0$ as $y \to 0$. *Hint: There are multiple ways of proving this, one way of proving this is to show that* $|e^{-2\pi|\xi|y} - 1| \leq 2\pi|\xi|y$ *first.*