Vrije Universiteit Amsterdam **20 October 2020 15:30-17:30**
Midterm Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**2+10+8+7+8 points**] Consider the 2π-periodic function f which is given by $f(x) = \pi^2 - x^2$ for $x \in [-\pi, \pi]$.

a) Draw a sketch of the function f on the interval $[-3\pi, 3\pi]$.

b) Show that the Fourier coefficients $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ of f are given by

$$\hat{f}(n) = \begin{cases} \frac{2\pi^2}{3} & n = 0 \\ \frac{-2(-1)^n}{n^2} & n \neq 0. \end{cases}$$

c) Show that

$$f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

for $x \in [-\pi, \pi]$.

d) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

e) Use Parseval's identity to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Question 2. [**2+2+2+2+2 points**] Answer the following questions with true or false: You do not have to explain your answers.

a) The space $\ell^2(\mathbb{Z})$ of square summable sequences with its usual norm is complete.

b) The space $\mathcal{R}$ of integrable functions on the circle with its usual norm is complete.

c) If a series is summable in the Cesàro sense, it is also summable in ordinary sense.

d) If a series is summable in the ordinary sense, it is summable in the Cesàro sense.

e) The Fourier series of a continuous function always converges to the function.

Question 3. [**5+5+10+10 points**] Let $0 \leq r < 1$ and $\theta \in [-\pi, \pi]$ and consider the Poisson kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

You are going to show that the Poisson kernel is a good kernel.

a) Show that the sum above converges absolutely (for $0 \leq r < 1$).

b) Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1.$$

c) Show that

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

Hint: Write $P_r(\theta) = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega^n}$ *for suitable* ω.

Note that

$$1 - 2r\cos(\theta) + r^2 = (1-r)^2 + 2r(1 - \cos(\theta))$$

from which it follows that $P_r(\theta) \geq 0$.

d) Let $0 < \delta < \pi$ and $s > 0$. Show that there exists a constant $C > 0$ such that for $\delta \leq |\theta| \leq \pi$ and $s \leq r < 1$

$$P_r(\theta) \leq C(1 - r^2).$$

Show that

$$\lim_{r \to 1} \int_{\delta \leq |\theta| \leq \pi} P_r(\theta) d\theta = 0.$$

Question 4. [**8+7 points**] In the book we have studied the pointwise convergence of Fourier series at points of continuity. Let $F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$ be the Fejer kernel and f a function on the circle. Recall that

$$f(x) = \lim_{N \to \infty} f * F_N(x) = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) F_N(x - y) dy,$$

whenever f is continuous at x. In this question we treat the situation where a function has a jump discontinuity. Let $f_1, f_2 : [-\pi, \pi] \to \mathbb{C}$ be two functions that are even, i.e. $f_1(-x) = f_1(x)$ and $f_2(-x) = f_2(x)$ for all $x \in [-\pi, \pi]$, and which are continuous at zero.

a) Show that

$$f_1(0) = \lim_{N \to \infty} \frac{1}{\pi} \int_{-\pi}^{0} f_1(y) F_N(-y) dy \quad \text{and} \quad f_2(0) = \lim_{N \to \infty} \frac{1}{\pi} \int_{0}^{\pi} f_2(y) F_N(-y) dy.$$

Let g be the function on the circle defined by

$$g(x) = \begin{cases} f_1(x) & -\pi \leq x \leq 0 \\ f_2(x) & 0 < x < \pi. \end{cases}$$

b) Show that

$$\lim_{N \to \infty} g * F_N(0) = \frac{1}{2}(f_1(0) + f_2(0)).$$