Vrije Universiteit Amsterdam **3 February 2020 18:30-21:15.**

Resit Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Recall that the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by the formula $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$, and that the Fourier coefficients of a 2π-periodic function g are given by $\widehat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx$.

Exercise 1. [**10+5+7+8 Points**] Let f be the 2π-periodic function $f(x) = |\sin(x)|$.

a) Show that the Fourier coefficients of f are given by

$$\widehat{f}(n) = \begin{cases} \frac{2}{\pi} & n = 0 \\ 0 & n \text{ is odd} \\ \frac{-2}{\pi}\frac{1}{n^2-1} & \text{otherwise.} \end{cases}$$

b) Show that the Fourier series of f converges uniformly to f.

c) Show that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}.$$

d) Use Parseval's identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

Exercise 2. [**8+7 Points**] Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function with Fourier coefficients $\widehat{f}$. Suppose that f is k times differentiable and that the k-th derivative $f^{(k)}$ is continuous. We assume that $k > 0$.

a) Show that $\widehat{f^{(k)}}(n) = (in)^k\widehat{f}(n)$.

b) Show that $\widehat{f}(n) = O(1/n^k)$ as $n \to \infty$.

continued on the back $\Longrightarrow$ ...

Exercise 3. [3+7+5 Points] Let $f \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\widehat{f}$.

a) Give the definition of the Schwarz space $\mathcal{S}(\mathbb{R})$.

b) Show that if $f \in \mathcal{S}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ that then $fg \in \mathcal{S}(\mathbb{R})$.

c) Suppose that f is an even function ($f(x) = f(-x)$ for all $x \in \mathbb{R}$). Show that the function $g(x) = (f(x))^2 + \widehat{f} * \widehat{f}(x)$ is its own Fourier transform.

Exercise 4. [7+7+6+10 Points]
Consider the differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + 4\pi i \frac{\partial u}{\partial x}(x,t) - 4\pi^2 u(x,t), \tag{1}$$

which we want to solve for $t > 0$. We will impose the initial value

$$u(x,0) = f(x)$$

for some function $f \in \mathcal{S}(\mathbb{R})$.

a) Take (formally) the Fourier transform of u in the x variable. Show that this Fourier transform $\widehat{u}$ satisfies
$$\frac{\partial \widehat{u}}{\partial t}(\xi,t) = -4\pi^2(\xi+1)^2 \widehat{u}(\xi,t). \tag{2}$$

b) Show that
$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2(\xi+1)^2 t} e^{2\pi i \xi x} d\xi$$
solves Equation (1). You are allowed to interchange derivatives and integrals without justification.

The function $x \mapsto u(x,t)$ is a Schwarz function for every $t > 0$, which you may use without proof below.

c) Use Plancherel's formula to show that
$$\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 |e^{-4\pi^2(\xi+1)^2 t} - 1|^2 d\xi.$$
for all $t > 0$.

d) Prove carefully that
$$\lim_{t \to 0} \int_{-\infty}^{\infty} |u(x,t) - f(x)|^2 dx = 0.$$
hint: Use part c) and split the integral in two parts: one part $\int_{|x| \geq N}$ *and one part* $\int_{|x| \leq N}$. *Choose* N *wisely.*