Vrije Universiteit Amsterdam **20 December 2019 12:00-14:00**
Final Fourier Analysis **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$. *The Fourier transform of a function $f \in \mathcal{M}(\mathbb{R})$ is defined by the formula $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi x i \xi}dx$.*

Question 1. [**5+10+5+5+10 Points**]
Let $g(x) = e^{-2\pi|x|}$ and $f(x) = xe^{-2\pi|x|}$. You may use that $f, g \in \mathcal{M}(\mathbb{R})$.

a) Show that $\hat{g}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$.

b) Show that $\hat{f}(\xi) = -\frac{i\xi}{\pi^2(1+\xi^2)^2}$. *Hint: You may use the transformation rules. These are valid for f, which you do not have to prove.*

c) Give the definition of the space $\mathcal{M}(\mathbb{R})$ and show that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

d) Show that for $x > 0$

$$xe^{-2\pi x} = \frac{2}{\pi^2}\int_0^{\infty} \frac{\xi \sin(2\pi x \xi)}{(1+\xi^2)^2}d\xi.$$

e) Show that

$$\int_{-\infty}^{\infty} \frac{\xi^2}{(1+\xi^2)^4}d\xi = \frac{\pi}{16}.$$

Question 2. [**8+7+5+5 Points**] Let $h \in \mathcal{S}(\mathbb{R})$. Consider the differential equation

$$-\frac{1}{4\pi}u''(x) + \pi u(x) = h(x). \tag{1}$$

a) Suppose that $u \in \mathcal{S}(\mathbb{R})$ solves (1). Show that the Fourier transform $\hat{u}$ then solves the equation

$$(\xi^2 + 1)\hat{u}(x) = \frac{1}{\pi}\hat{h}(\xi).$$

b) Give the definition of the Schwartz space $\mathcal{S}(\mathbb{R})$ and show that the function k, given by $k(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}\hat{h}(\xi)$ is an element of the Schwarz space $\mathcal{S}(\mathbb{R})$.

c) Let g be the function in Question 1. Let $u(x) = g * h(x)$. Show that $u \in \mathcal{S}(\mathbb{R})$.

d) Show that $u = g * h$ solves (1).

Question 3. [**10 Points**] Let $f, g \in \mathcal{S}(\mathbb{R})$ be given by $g(x) = \cos(2\pi x)f(x)$. Compute the Fourier transform $\widehat{g}$ in terms of the Fourier transform $\widehat{f}$.

Question 4. [**5+15 Points**] Let $L > 0$. Let $f \in \mathcal{S}(\mathbb{R})$ be a function such that the Fourier transform $\hat{f}$ satisfies $\hat{f}(\xi) = 0$ if $|\xi| \geq L$ (The Fourier transform has support in $[-L, L]$).

a) State Plancherel's formula.

b) Show that $\int_{-\infty}^{\infty} |\frac{df}{dx}(x)|^2 dx \leq 4\pi^2 L^2 \int_{-\infty}^{\infty} |f(x)|^2 dx$. *Hint: Use Plancherel's formula.*