Vrije Universiteit Amsterdam | **22 October 2019 8:45-10:45**
Midterm Fourier Analysis | **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Exercise 1. [**2+10+10+5+8 points**] Let $c > 0$ and let f be the 2π-periodic function, such that for $-\pi \le x < \pi$ we have

$$f(x) = e^{cx} + e^{-cx}.$$

a) Make a sketch of the graph of f on the interval $[-4\pi, 4\pi]$.

b) Show that the Fourier coefficients $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ are

$$\hat{f}(n) = \frac{c}{\pi} \cdot \frac{(-1)^n}{c^2+n^2}(e^{c\pi} - e^{-c\pi}).$$

c) Show that the Fourier series of f converges uniformly to f.

d) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{c^2+n^2} = \frac{\pi}{c} \cdot \frac{e^{c\pi}+e^{-c\pi}}{e^{c\pi}-e^{-c\pi}}.$$

e) Use Parseval's identity to compute

$$\sum_{n=-\infty}^{\infty} \frac{1}{(c^2+n^2)^2}.$$

Exercise 2. [**10 points**] Let f be an integrable function on the circle, such that

$$f(\theta+\pi) = -f(\theta)$$

for all θ. Show that

$$\hat{f}(n) = 0,$$

when n is even.

Continued on the back $\Rightarrow \ldots$

Exercise 3. [**12+13+10 points**] Let $f_k : [-\pi, \pi] \to \mathbb{C}$ be a sequence of functions. Assume that each f_k is continuously differentiable and that $f_k' \to g$ uniformly as $k \to \infty$. Let $a \in [-\pi, \pi]$, $c \in \mathbb{C}$ and assume that $f_k(a) \to c$ as $k \to \infty$.

a) Show that f_k converges uniformly to a function $f : [-\pi, \pi] \to \mathbb{R}$ with $f' = g$. *Hint: use the fact that* $f_k(x) = f_k(a) + \int_a^x f_k'(y)dy$.

b) Let c_n be a sequence of complex numbers such that $\sum_{n=-\infty}^{\infty} |nc_n| < \infty$. Show that the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad g(x) = \sum_{n=-\infty}^{\infty} inc_n e^{inx}$$

converge uniformly and that f is continuously differentiable with $f'(x) = g(x)$.

The function $g : [-\pi, \pi] \to \mathbb{C}$ with $g(x) = x^2$ has a uniformly converging Fourier series. From this we can derive the identity

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{n^2} e^{inx} = \frac{x^2}{2} - \frac{\pi^2}{6} \qquad \text{for} \qquad -\pi \leq x \leq \pi.$$

(You do not have to prove this).

c) Express

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx) \qquad \text{for} \qquad -\pi \leq x \leq \pi,$$

as a polynomial and evaluate at $x = \frac{\pi}{2}$ to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Exercise 4. [**5+5 points**] Let c_k be a sequence of complex numbers.

a) State the definition of (ordinary) convergence of the series $\sum_{k=0}^{\infty} c_k$ to s.

b) State the definition of Cesàro summability of $\sum_{k=0}^{\infty} c_k$ to σ.