Exam Empirical Methods

VU University Amsterdam, Faculty of Exact Sciences

18.30 – 21.15h, February 12, 2015

- Question 1 is on this page.
- Always *motivate* your answers.
- Write your answers in English.
- Only the use of a simple, non-graphical calculator is allowed.
- Programmable/graphical calculators, laptops, mobile phones, smart watches, books, own formula sheets, etc. are not allowed.
- On the last four pages of the exam, some formulas and tables that you may want to use can be found.
- The total number of points you can receive is 90: Grade $= 1 + \dfrac{\text{points}}{10}$.
- The division of points per question and subparts is as follows:

Question	1	2	3	4	5	6	7
Part a)	3	3	4	2	8	4	2
Part b)	4	3	3	5	2	2	2
Part c)	4	3	8	2	2	2	6
Part d)	3	2	3	2	-	6	-
Total	14	11	18	11	12	14	10

- If you are asked to perform a test, do not only give the conclusion of your test, but report:
  1. the hypotheses in terms of the population parameter of interest;
  2. the significance level;
  3. the test statistic and its distribution under the null hypothesis;
  4. the observed value of the test statistic;
  5. the P-value or the critical value(s);
  6. whether or not the null hypothesis is rejected and why;
  7. finally, phrase your conclusion in terms of the context of the problem.

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1. Alice throws a fair coin twice.
   (a) Give the sample space Ω and probability measure P for this experiment.
   (b) Are the events $A = \{\text{First throw is Heads}\}$ and $B = \{\text{Precisely one throw is Tails}\}$ independent events?
   (c) Alice receives 2 euros for each time she throws Heads, but she loses 1 euro for each time she throws Tails. Let X be the random variable which denotes the amount Alice earns after the two throws.
   What is the probability distribution of X?
   (d) Compute, using part (c), the expectation $E(X)$ of X.

2. Figure 1 below shows a boxplot and a normal Q-Q plot of a sample x.

   (a) Describe briefly what the boxplot tells you about the location, spread and shape of the underlying distribution of the data.

   (b) What can you deduce from the Q-Q plot with respect to the tails of the underlying distribution of the data compared to the tails of a normal distribution?

   (c) For each of the histograms in Figure 2 below indicate why it could or could not be a histogram of the sample x.

   (d) Will the sample mean of x be larger, smaller or roughly equal to the sample median?

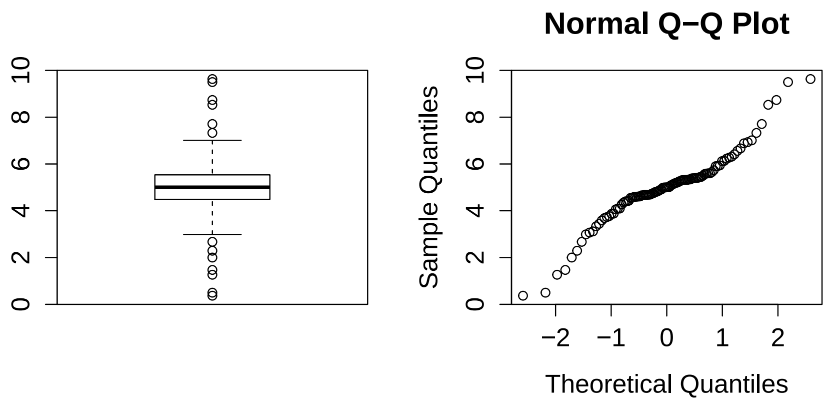


Figure 1: Boxplot and normal Q-Q plot of a sample x.

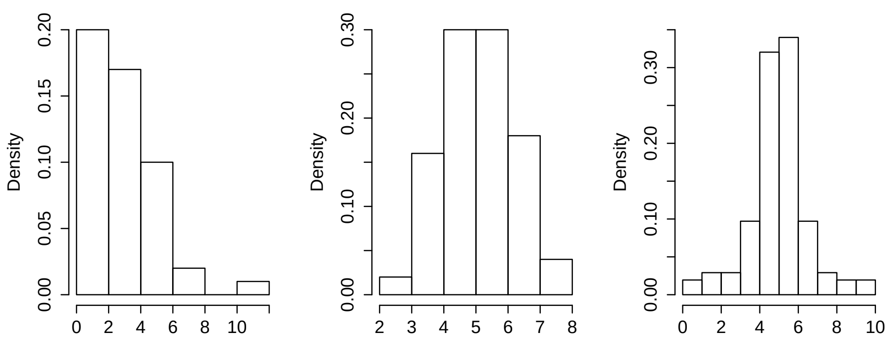


Figure 2: Three histograms.

3. Assume that the amount of beer in a randomly selected beer bottle has a normal distribution with mean $\mu = 300$ ml and standard deviation $\sigma = 5$ ml.

   (a) What is the probability that one randomly selected beer bottle contains between 294 and 307 ml of beer?

   (b) What is the probability that the mean volume of beer in a random sample of $n = 25$ beer bottles is at least 299 ml?

   Now assume the amount of beer in a beer bottle from company A is normally distributed with unknown mean μ and known standard deviation $\sigma = 5$ ml. The amount of beer in $n = 16$ randomly selected beer bottles from company A is measured and the sample mean equals $\overline{x} = 298.2$

(c) Use the P-value method for a suitable hypothesis test (motivate your choice!) to test the claim that the mean amount of beer is less than 300 ml at significance level $\alpha = 0.05$.

(d) If a 90% confidence interval with margin of error $E = 1$ ml were required for the mean amount of beer in a bottle, how many bottles should be measured?

4. On a particular day, 22 out of 542 visitors to a website clicked on a certain web banner. After the banner was modified, it was found that 64 out of 601 visitors to the website on a day clicked on the web banner.

   (a) Give a point estimate for the difference between the proportions of people who click on the banner before and after the modification.

   (b) Construct a 95% confidence interval for the difference between the proportions of people who click on the banner before and after the modification.

   (c) What is the interpretation of the confidence interval obtained in part (b)?

   (d) Based on your answer of part (b), was the modification successful?

5. A researcher wants to investigate the claim that among married couples, females speak more words in a day than males. She randomly selects 71 couples and the total number of words spoken in a day is counted for both the husband (sample 1) and wife (sample 2). Some sample statistics regarding this experiment which you may or may not use in your analysis are shown below ($\overline{d}$ and s_d denote the mean and standard deviation of the pairwise differences in total number of words spoken between husband and wife and s_p denotes the pooled sample standard deviation):

$$\overline{x}_1 = 16576.1, \overline{x}_2 = 18443.3, \overline{d} = -1867.2,$$
$$s_1 = 7871.5, s_2 = 7459.6, s_d = 8955.2, s_p = 7668.3.$$

   (a) Test with a suitable hypothesis test (motivate your choice!) the claim that among married couples, females speak more words in a day than males. Take significance level $\alpha = 0.05$.

   (b) The test you performed in part (a) should only be used if certain requirement(s) are met. What are these requirement(s) and are they met in this case?

   (c) What is the interpretation of the significance level $\alpha = 0.05$?

6. Estimating the costs of drilling oil wells is an important consideration for the oil industry. For 16 randomly selected oil wells both their depth (km) and drilling costs (mln EUR) were measured and stored in respective datasets x and y. A linear regression analysis was carried out with explanatory variable 'depth' and response variable 'drilling costs'. Some sample statistics of the data that you may or may not use are:

$$\overline{x} = 2.58, \overline{y} = 6.35, s_x = 0.80, s_y = 2.80, r = 0.95, \sqrt{\frac{1-r^2}{n-2}} = 0.081, s_{b_0} = 0.77, s_{b_1} = 0.28.$$

   Furthermore, a scatterplot of the drilling costs against the depth of oil wells is shown in Figure 3 (see next page).

(a) Provide an estimate for the regression equation by eye.

(b) Based on your answer of part (a), what is your prediction for the drillings costs of a well with a depth of 4.0 km?

(c) Compute the coefficient of determination. What is its interpretation?

(d) Test the claim that $\rho = 0$, i.e. that there is no linear correlation between depth and drilling costs of oil wells. Take significance level $\alpha = 0.05$.

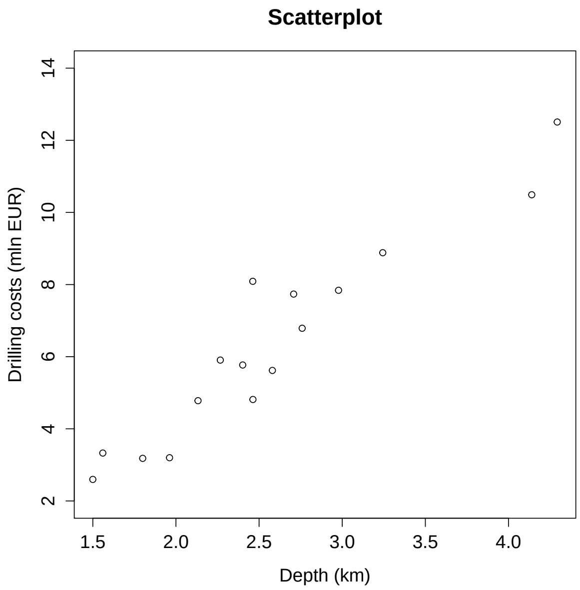


Figure 3: Scatterplot of drilling costs against depth of oil wells.

7. A variety of different datasets includes numbers with leading (first) digits that follow, according to Benford's law, the following distribution:

Leading digit	1	2	3	4	5	6	7	8	9
Percentage	30.1%	17.6%	12.5%	9.7%	7.9%	6.7%	5.8%	5.1%	4.6%

Since the numbers people report in tax files are among the datasets that should behave according to Benford's law, this law can be used to detect fraud: if the observed frequencies of the leading digits differ significantly from the expected frequencies according to Benford's law, then the tax file appears to result from fraud. A tax inspector checks a tax file with 377 numbers and finds the following frequencies of leading digits:

Leading digit	1	2	3	4	5	6	7	8	9
Frequency	132	61	51	43	32	25	18	11	4

(a) Compute the expected frequency of 9 as leading digit for this tax file under the assumption that the leading digits follow the distribution specified by Benford's law.

(b) Use part (a) to show that the requirements for a chi-square goodness-of-fit test are satisfied.

(c) Perform a chi-square goodness-of-fit test to investigate whether the tax file appears to be legitimate. Use significance level $\alpha = 0.01$.
The observed value for the test statistic is 19.87, so you do not have to compute this value!

Formulas and Tables for Exam Empirical Methods

Probability

We use the following notation:
Ω sample space, P probability measure.
$B, A_1, A_2, \ldots, A_m$ events,
$A_1, A_2, \ldots, A_m$ a partition of Ω with $P(A_i) > 0$ for all $i \in \{1, 2, \ldots, m\}$.

Law of Total Probability:

$$P(B) = \sum_{i=1}^{m} P(B \cap A_i) = \sum_{i=1}^{m} P(B|A_i)P(A_i).$$

Bayes' Theorem:

$$P(A_r|B) = \frac{P(A_r \cap B)}{\sum_{i=1}^{m} P(B|A_i)P(A_i)} = \frac{P(B|A_r)P(A_r)}{\sum_{i=1}^{m} P(B|A_i)P(A_i)}.$$

Two *independent* samples

(The statements below hold if certain requirements are met.)

For two *independent* samples,
(i) if σ_1 and σ_2 are unknown and $\sigma_1 \neq \sigma_2$, the test statistic

$$T_2 = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

has a t-distribution with approximately $\tilde{n}$ degrees of freedom under the null hypothesis. We use the conservative estimate $\tilde{n} = \min\{n_1 - 1, n_2 - 1\}$.
(ii) if σ_1 and σ_2 are unknown and $\sigma_1 = \sigma_2$, then the test statistic

$$T_2^{\text{eq}} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2/n_1 + s_p^2/n_2}}$$

has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom under the null hypothesis. Here s_p is the square root of the pooled sample variance s_p^2 given by

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

(iii) if σ_1 and σ_2 are known, then the test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

has a standard normal distribution under the null hypothesis.
(iv) if $p_1 = p_2$, the test statistic

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}(1-\bar{p})/n_1 + \bar{p}(1-\bar{p})/n_2}}$$

approximately has a standard normal distribution. Here $\bar{p} = (x_1 + x_2)/(n_1 + n_2)$ is the pooled sample proportion.
(v) the margin of error for a $1 - \alpha$ confidence interval for $p_1 - p_2$ is given by

$$E = z_{\alpha/2}\sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}.$$

Correlation

Under certain conditions the test statistic

$$T_{cor} = \frac{r - \rho}{\sqrt{(1-r^2)/(n-2)}}$$

has a t-distribution with $n-2$ degrees of freedom. Here ρ is the population linear correlation coefficient and r is the sample linear correlation coefficient given by

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\Big[\frac{(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y}\Big].$$

Linear regression

Let β_0 be the unknown intercept and β_1 the unknown slope of a linear regression model with one explanatory variable, and let b_0 and b_1 be the corresponding estimators, i.e. the intercept and slope of the regression line (the 'best' line). Then b_0 and b_1 are given by

$$b_1 = r\frac{s_y}{s_x}$$

and

$$b_0 = \bar{y} - b_1\bar{x}.$$

If certain requirements are met, then the test statistic

$$T_1 = \frac{b_1 - \beta_1}{s_{b_1}}$$

has a t-distribution with $n-2$ degrees of freedom. Here s_{b_1} is the standard error (i.e. estimated standard deviation) of the estimator b_1.

NEGATIVE z Scores

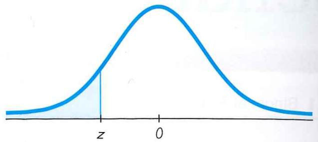


Table 2 Standard Normal (z) Distribution: Cumulative Area from the LEFT

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.50 and lower	.0001									
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051 *	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505 *	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

NOTE: For values of *z* below 3.49, use 0.0001 for the area.
*Use these common values that result from interpolation:

z Score	Area
−1.645	0.0500
−2.575	0.0050

(continued)

POSITIVE z Scores

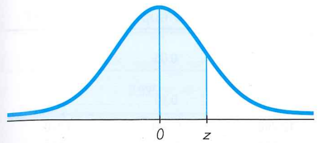


Table 2 *(continued)* Cumulative Area from the LEFT

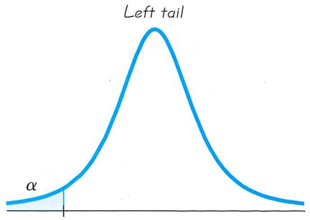
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495 *	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949 *	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.50 and up	.9999									

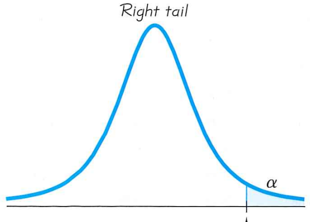
NOTE: For values of *z* above 3.49, use 0.9999 for the area.
*Use these common values that result from interpolation:

z Score	Area
1.645	0.9500
2.575	0.9950

Common Critical Values

Confidence Level	Critical Value
0.90	1.645
0.95	1.96
0.99	2.575





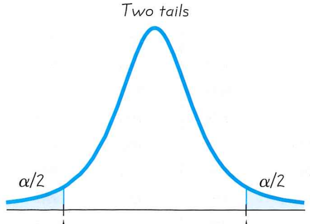


Table 3 *t* Distribution: Critical *t* Values

	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
	Area in Two Tails				
Degrees of Freedom	0.01	0.02	0.05	0.10	0.20
1	63.657	31.821	12.706	6.314	3.078
2	9.925	6.965	4.303	2.920	1.886
3	5.841	4.541	3.182	2.353	1.638
4	4.604	3.747	2.776	2.132	1.533
5	4.032	3.365	2.571	2.015	1.476
6	3.707	3.143	2.447	1.943	1.440
7	3.499	2.998	2.365	1.895	1.415
8	3.355	2.896	2.306	1.860	1.397
9	3.250	2.821	2.262	1.833	1.383
10	3.169	2.764	2.228	1.812	1.372
11	3.106	2.718	2.201	1.796	1.363
12	3.055	2.681	2.179	1.782	1.356
13	3.012	2.650	2.160	1.771	1.350
14	2.977	2.624	2.145	1.761	1.345
15	2.947	2.602	2.131	1.753	1.341
16	2.921	2.583	2.120	1.746	1.337
17	2.898	2.567	2.110	1.740	1.333
18	2.878	2.552	2.101	1.734	1.330
19	2.861	2.539	2.093	1.729	1.328
20	2.845	2.528	2.086	1.725	1.325
21	2.831	2.518	2.080	1.721	1.323
22	2.819	2.508	2.074	1.717	1.321
23	2.807	2.500	2.069	1.714	1.319
24	2.797	2.492	2.064	1.711	1.318
25	2.787	2.485	2.060	1.708	1.316
26	2.779	2.479	2.056	1.706	1.315
27	2.771	2.473	2.052	1.703	1.314
28	2.763	2.467	2.048	1.701	1.313
29	2.756	2.462	2.045	1.699	1.311
30	2.750	2.457	2.042	1.697	1.310
31	2.744	2.453	2.040	1.696	1.309
32	2.738	2.449	2.037	1.694	1.309
33	2.733	2.445	2.035	1.692	1.308
34	2.728	2.441	2.032	1.691	1.307
35	2.724	2.438	2.030	1.690	1.306
36	2.719	2.434	2.028	1.688	1.306
37	2.715	2.431	2.026	1.687	1.305
38	2.712	2.429	2.024	1.686	1.304
39	2.708	2.426	2.023	1.685	1.304
40	2.704	2.423	2.021	1.684	1.303
45	2.690	2.412	2.014	1.679	1.301
50	2.678	2.403	2.009	1.676	1.299
60	2.660	2.390	2.000	1.671	1.296
70	2.648	2.381	1.994	1.667	1.294
80	2.639	2.374	1.990	1.664	1.292
90	2.632	2.368	1.987	1.662	1.291
100	2.626	2.364	1.984	1.660	1.290
200	2.601	2.345	1.972	1.653	1.286
300	2.592	2.339	1.968	1.650	1.284
400	2.588	2.336	1.966	1.649	1.284
500	2.586	2.334	1.965	1.648	1.283
1000	2.581	2.330	1.962	1.646	1.282
2000	2.578	2.328	1.961	1.646	1.282
Large	2.576	2.326	1.960	1.645	1.282

Table 4 Chi-Square (χ^2) Distribution

	Area to the Right of the Critical Value									
Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.299
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

Source: Donald B. Owen, *Handbook of Statistical Tables.*

Degrees of Freedom

$n - 1$	**Confidence interval or hypothesis test** for a standard deviation σ or variance σ^2
$k - 1$	**Goodness-of-fit** with *k* categories
$(r - 1)(c - 1)$	**Contingency table** with *r* rows and *c* columns
$k - 1$	**Kruskal-Wallis test** with *k* samples