# Econometrics III: Example Exam

## Problem 1 (25/60 points)

Suppose a model for an industrial sector's profit growth ($y_1$) and investment growth ($y_2$) is given by the bivariate vector autoregressive (VAR) process for $\mathbf{y}_t = (y_{1t}, y_{2t})'$:

$$\mathbf{y}_t = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} \mathbf{y}_{t-1} + \mathbf{u}_t, \quad t = 0, \pm 1, \pm 2, ... \tag{1}$$

The innovations $\mathbf{u}_t = (u_{1t}, u_{2t})'$ are assumed to be generated by an uncorrelated Gaussian process with mean zero and fixed, nonsingular covariance matrix,

$$\Sigma_{\mathbf{u}} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix}.$$

(a) (4 points) Show that the process (1) is stable.

(b) (4 points) Compute the unconditional mean of $y_{1t}$ (profit growth).

(c) (3 points) Does profit growth Granger-cause investment growth? Does investment growth Granger-cause profit growth? Justify briefly.

(d) (4 points) Compute the impulse responses of the VAR given in (1) for horizons $h = 0, 1, 2$. Given a unit shock to profit growth in $t = 0$, what effect on investment growth do you expect as $h \to \infty$?

(e) (2 points) If you wanted to consider orthogonalized impulse responses, would they be different from the ones you computed in (d)? Why or why not?

(f) (8 points) Suppose that at time $T$, the observed values $\mathbf{y}_T = (y_{T1}, y_{T2})'$ are $(0.4, 0)'$.

 (i) Compute the 1-step and the 2-step-ahead predictions $\mathbf{y}_T(1)$ and $\mathbf{y}_T(2)$.

 (ii) Compute the forecast error variance matrix for the 2-step ahead prediction of $\mathbf{y}_t$.

 (iii) Compute a 95% prediction interval for the 2-step ahead prediction of $y_1$ (profit growth).

## Problem 2 (20/60 points)

Suppose you want to analyze a time series data set on three exchange rate series $y_1$, $y_2$ and $y_3$. Unit root tests have shown that all three time series are $I(1)$.

(a) (8 points) Economic theory suggests that two equilibrium relationships exist between the three variables, implying that $y_1 - 0.3y_2$ is stationary and $y_1 - 2y_3$ is stationary.

 (i) Set up a zero-mean vector-error correction model without lagged differences for the vector $\mathbf{y}_t = (y_{1t}, y_{2t}, y_{3t})'$, incorporating the information from economic theory. Give the dimensions of all model components.

 (ii) A common normalization for the cointegration matrix is $\beta' = (\mathrm{I}_r : \tilde{\beta}'_{K-r})$, where $r$ is the number of cointegration relations, $\mathrm{I}_r$ is the identity matrix of dimension $r$, and $K$ is the number of variables in the system.
 $\widetilde{\beta}_{K-r}$ is a $((K-r) \times r)$ matrix of coefficients. Rewrite the VECM you set up in part (i) in terms of this normalization. *Hint: Consider the ordering of the variables.*

(b) (8 points) Suppose you believe the economic theory regarding $\beta$, but you want to estimate the loading matrix $\alpha$ of your VECM in part (a). Suppose $T$ observations plus a suitable number of pre-sample observations of $\mathbf{y}_t$ are available. Define

$$\begin{aligned} \Delta\mathbf{Y} &= (\Delta\mathbf{y}_1 : ... : \Delta\mathbf{y}_T), \\ \mathbf{Y}_{-1} &= (\mathbf{y}_0 : ... : \mathbf{y}_{T-1}), \\ \mathbf{U} &= (\mathbf{u}_1 : ... : \mathbf{u}_T) \end{aligned}$$

Write down the model in matrix notation. Show that the least squares estimator for the loading matrix $\alpha$ can be expressed as

$$\widehat{\alpha} = \Delta\mathbf{Y}\mathbf{Y}_{-1}'\beta(\beta'\mathbf{Y}_{-1}\mathbf{Y}_{-1}'\beta)^{-1} \tag{2}$$

(c) (4 points) Suppose the number of cointegration relations for the three variables is unknown and you want to find out about it using a test. State an suitable test sequence by mentioning the test type, null- and alternative hypotheses and the decision rule(s).

## Problem 3 (15/60 points)

Consider the one-way error component panel data model for a sample on $N$ individuals at $T$ time points in matrix notation:

$$y = X\beta + G\mu + e \quad e|X \sim N(0, \sigma_e^2 \mathrm{I}_{NT}) \tag{3}$$

where

- $\beta = (\beta_1, ..., \beta_K)'$ is a vector of unknown fixed coefficients,
- $\mu$ is a $(N \times 1)$-vector of unknown fixed individual-specific intercepts,
- $X$ is a $(NT \times K)$-matrix of strictly exogenous random regressors,
- $y$ and $e$ are $(NT \times 1)$-vectors of dependent variable observations and disturbances, respectively,
- $G = \mathrm{I}_N \otimes \mathbb{1}_T$ is a matrix of individual dummies. $\mathbb{1}_T$ denotes a $(T \times 1)$-vector of ones and $\mathrm{I}_N$ denotes the identity matrix of dimension $N$.

(a) (7 points) Define the matrices $P_G := G(G'G)^{-1}G'$ and $M_G := \mathrm{I}_{NT} - P_G$. Transform model (3) by pre-multiplying both sides of the equation by $M_G$. What are the properties of the transformed data vector $M_G y$ and the transformed regressor matrix $M_G X$? Do you consider the OLS estimator in this transformed model a good choice to estimate the unknown parameter vector $\beta$? Explain briefly.

(b) (8 points) Derive the conditional expectation and the conditional covariance matrix of the within estimator

$$\widehat{\beta}_W = (X'M_G X)^{-1}X'M_G y, \tag{4}$$

where $M_G$ is given in part (a), conditional on the regressor matrix $X$.

# Appendix

**Rules for Kronecker product and vectorization (vec) operator**

Assume that the matrices A, B, C, and I (identity matrix) have dimensions such that all products exist.

(K1) $(A \otimes B)' = A' \otimes B'$

(K2) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(K3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

(K4) $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$

(K4*) $\text{vec}(AB) = (\text{I} \otimes A)\text{vec}(B) = (B' \otimes \text{I})\text{vec}(A)$

(K5) $\text{vec}(B')'\text{vec}(A) = \text{vec}(A')'\text{vec}(B) = \text{tr}(BA) = \text{tr}(AB)$

(K6) $\text{tr}(ABC) = \text{vec}(A')'(\text{I} \otimes B)\text{vec}(C)$