Econometrics III: Solution to Example Exam

Solution to problem 1

(a) To check stability, check whether all roots of the reverse characteristic polynomial are larger than 1 in modulus.

$$det(\mathrm{I} - Az) = det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0.8z & 0.2z\\ 0 & 0.5z\end{pmatrix}\right) = (1-0.8z)(1-0.5z) \tag{1}$$

Therefore $1 - 0.8z = 0 \Leftrightarrow z = 1.25$ or $1 - 0.5z = 0 \Leftrightarrow z = 2$.

(b) Unconditional mean:

$$\mathbb{E}[\mathbf{y}_t] = \left(\mathrm{I}_2 - \begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix}\right)^{-1} \cdot \begin{pmatrix}0.2\\ 0.1\end{pmatrix} \tag{2}$$

$$= \frac{1}{0.1}\begin{pmatrix}0.5 & 0.2\\ 0 & 0.2\end{pmatrix} \cdot \begin{pmatrix}0.2\\ 0.1\end{pmatrix} = \begin{pmatrix}1.2\\ 0.2\end{pmatrix} \tag{3}$$

(c) Profit growth (y_1) does not Granger-cause investment growth (y_2), because $a_{21,1} = 0$.
Investment growth (y_2) Granger-causes profit growth (y_1), because $a_{12,1} = 0.2 \neq 0$.

(d) $\Phi_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\Phi_1 = \mathbf{A}_1 = \begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix}$ and

$$\Phi_2 = \mathbf{A}_2 = \mathbf{A}_1^2 = \begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix}\begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix} = \begin{pmatrix}0.64 & 0.26\\ 0 & 0.25\end{pmatrix}$$

A unit shock to profit growth has no impact on investment growth for any h. Reason: no Granger causality; impulse responses are zero.

(e) No, because the shocks are not contemporaneously correlated.

(f) (i)

$$\mathbf{y}_T(1) = \begin{pmatrix}0.2\\ 0.1\end{pmatrix} + \begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix}\begin{pmatrix}0.4\\ 0\end{pmatrix} = \begin{pmatrix}0.52\\ 0.1\end{pmatrix} \tag{4}$$

$$\mathbf{y}_T(2) = \begin{pmatrix}0.2\\ 0.1\end{pmatrix} + \begin{pmatrix}0.8 & 0.2\\ 0 & 0.5\end{pmatrix}\begin{pmatrix}0.52\\ 0.1\end{pmatrix} = \begin{pmatrix}0.636\\ 0.15\end{pmatrix} \tag{5}$$

(ii)

$$MSE[\mathbf{y}_T(2)] = \sum_{i=0}^{1}\Phi_i\Sigma_{\mathbf{u}}\Phi_i' = \Sigma_{\mathbf{u}} + \Phi_1\Sigma_{\mathbf{u}}\Phi_1' = \begin{pmatrix}0.3 & 0\\ 0 & 0.6\end{pmatrix} + \begin{pmatrix}0.216 & 0.06\\ 0.06 & 0.15\end{pmatrix} = \begin{pmatrix}0.516 & 0.06\\ 0.06 & 0.75\end{pmatrix} \tag{6}$$

(iii) 95% prediction interval:

$$\widehat{y}_{1,T}(2) \pm 1.96 \cdot \sqrt{MSE[\mathbf{y}_T(2)]_{11}} = 0.636 \pm 1.96 \cdot \sqrt{0.516} \approx [0.772, 2.044]$$

Solution to problem 2

(a) (i) Model:

$$\Delta\mathbf{y}_t = \alpha\beta'\mathbf{y}_{t-1} + \mathbf{u}_t, \tag{7}$$

with

$$\Delta\mathbf{y}_t = \begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \\ \Delta y_{3t} \end{pmatrix} \tag{8}$$

where $\mathbf{y}_t$, $\mathbf{y}_{t-1}$ and $\mathbf{u}_t$ are (3×1)-vectors and α and β are (3×2)-matrices. According to the economic theory, we have

$$\beta = \begin{pmatrix} 1 & 1 \\ -0.3 & 0 \\ 0 & -2 \end{pmatrix} \tag{9}$$

(ii) New model:

$$\Delta\widetilde{\mathbf{y}}_t = \alpha\beta'\widetilde{\mathbf{y}}_{t-1} + \mathbf{u}_t, \tag{10}$$

with

$$\Delta\widetilde{\mathbf{y}}_t = \begin{pmatrix} \Delta y_{2t} \\ \Delta y_{3t} \\ \Delta y_{1t} \end{pmatrix}, \tag{11}$$

and

$$\beta = \begin{pmatrix} -0.3 & 0 \\ 0 & -2 \\ 1 & 1 \end{pmatrix} \quad \Leftrightarrow \quad \widetilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{0.3} & -\frac{1}{2} \end{pmatrix} \tag{12}$$

(b) Model in matrix notation:

$$\Delta\mathbf{Y} = \Pi\mathbf{Y}_{-1} + \mathbf{U} = \alpha\beta'\mathbf{Y}_{-1} + \mathbf{U} \tag{13}$$

Vectorized model:

$$\mathrm{vec}(\Delta\mathbf{Y}) = (\mathbf{Y}'_{-1}\beta \otimes \mathrm{I}_K)\mathrm{vec}(\alpha) + \mathrm{vec}(\mathbf{U}) \tag{14}$$

OLS estimator:

$$\begin{aligned}
\mathrm{vec}(\widehat{\alpha}) &= \left[(\mathbf{Y}_{-1}\beta \otimes \mathrm{I}_K)'(\mathbf{Y}_{-1}\beta \otimes \mathrm{I}_K)\right]^{-1}(\mathbf{Y}_{-1}\beta \otimes \mathrm{I}_K)'\mathrm{vec}(\Delta\mathbf{y}) && (15)\\
&= (\beta'\mathbf{Y}_{-1}\mathbf{Y}'_{-1}\beta \otimes \mathrm{I}_K)^{-1}((\beta'\mathbf{Y}_{-1} \otimes \mathrm{I}_K)\mathrm{vec}(\Delta\mathbf{Y}) && (16)\\
&= \mathrm{vec}[\Delta\mathbf{Y}\mathbf{Y}'_{-1}\beta(\beta'\mathbf{Y}_{-1}\mathbf{Y}'_{-1}\beta)^{-1}] && (17)
\end{aligned}$$

(c) Test sequence:

1. H_0: $\mathrm{rank}(\Pi) = 0$ vs. H_1: $\mathrm{rank}(\Pi) > 0$
2. H_0: $\mathrm{rank}(\Pi) = 1$ vs. H_1: $\mathrm{rank}(\Pi) > 1$
3. H_0: $\mathrm{rank}(\Pi) = 2$ vs. H_1: $\mathrm{rank}(\Pi) = 3$

Decision rule: Terminate the test sequence once H_0 is not rejected for the first time.

Solution to problem 3

(a) Transformed model:

$$M_G y = M_G X\beta + \underbrace{M_G G}_{=0}\mu + M_G e$$

$M_G y$ is a $(NT \times 1)$-vector containing the differences between observations y_{it} and the individual-specific means of the dependent variable observations over time. The columns of the matrix $M_G X$ contain the differences between regressor observations and the individual-specific means of the regressor observations over time.
The OLS estimator/within estimator is a good choice to estimate this model, as it is efficient.

(b) Expectation:

$$\begin{aligned}
\mathbb{E}[\widehat{\beta}_W|X] &= \mathbb{E}[(X'M_GX)^{-1}X'M_Gy|X] && (18)\\
&= (X'M_GX)^{-1}X'M_G\mathbb{E}[X\beta + G\mu + e|X] && (19)\\
&= (X'M_GX)^{-1}X'M_G(M_GX\beta + \underbrace{M_GG}_{=0}\mu + M_G\underbrace{\mathbb{E}[e|X]}_{=0}) && (20)\\
&= (X'M_GX)^{-1}X'M_GX\beta && (21)\\
&= \beta && (22)
\end{aligned}$$

Covariance matrix:

$$\begin{aligned}
\mathrm{V}[\widehat{\beta}_W|X] &= \mathrm{V}[\beta + (X'M_GX)^{-1}X'M_Ge] && (23)\\
&= (X'M_GX)^{-1}X'M_G\underbrace{\mathrm{V}[e|X]}_{=\sigma_e^2\mathrm{I}_{NT}}M_GX(X'M_GX)^{-1} && (24)\\
&= \sigma_e^2(X'M_GX)^{-1} && (25)
\end{aligned}$$

Throughout, we have used the projection properties of M_G: $M_G = M_G' = M_GM_G$.