Solution Ectr. III Exam

Solution to problem 1

(a) Reverse characteristic polynomial:

$$\begin{aligned}\det(\mathrm{I}_K - \mathbf{A}_1 z) &= \det\begin{pmatrix} 1-(1+\rho)z & 0 \\ -\kappa z & 1-\delta z \end{pmatrix} \\ &= [1-(1+\rho)z]\cdot(1-\delta z) - 0\end{aligned}$$

Setting the two factors to zero gives roots

$$\begin{aligned} 1-(1+\rho)z = 0 \quad &\Leftrightarrow \quad z_1 = \frac{1}{1+\rho} \\ 1-\delta z = 0 \quad &\Leftrightarrow \quad z_2 = \frac{1}{\delta}\end{aligned}$$

For stability, we need $|z_1| > 1$ and $|z_2| > 1$. This is guaranteed if $-2 < \rho < 0$ and $|\delta| < 1$.

(b) Mean:

$$\mu = (\mathrm{I}_K - \mathbf{A}_1)^{-1}c = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1+\rho & 0 \\ \kappa & \delta \end{pmatrix}\right)^{-1}\begin{pmatrix} -\rho \\ 1-\delta-\kappa \end{pmatrix}$$

Compute inverse:

$$\begin{pmatrix} -\rho & 0 \\ -\kappa & 1-\delta \end{pmatrix}^{-1} = \frac{1}{-\rho(1-\delta)}\begin{pmatrix} 1-\delta & 0 \\ \kappa & -\rho \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho} & 0 \\ \frac{\kappa}{-\rho(1-\delta)} & \frac{1}{1-\delta} \end{pmatrix}$$

Therefore,

$$\mu = \begin{pmatrix} -\frac{1}{\rho} & 0 \\ \frac{\kappa}{-\rho(1-\delta)} & \frac{1}{1-\delta} \end{pmatrix}\begin{pmatrix} -\rho \\ 1-\delta-\kappa \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{\kappa}{\rho(1-\delta)}\cdot(-\rho) + \frac{1-\delta-\kappa}{1-\delta} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c) Inflation Granger-causes unemployment if $\kappa \neq 0$. Unemployment does not Granger-cause inflation, because $a_{12} = 0$.
These results do not change if $\delta = 0$. $\delta = 0$ only means that the model for unemployment does not contain an autoregressive term.

(d) The impulse responses are contained in the moving average coefficient matrices Φ_i. Since we have a $VAR(1)$ model, $\Phi_i = \mathbf{A}_1^i$:

$$\begin{aligned}\Phi_0 &= \mathrm{I}_2 \\ \Phi_1 &= \mathbf{A}_1 = \begin{pmatrix} 1+\rho & 0 \\ \kappa & \delta \end{pmatrix} \\ \Phi_2 &= \mathbf{A}_1^2 = \begin{pmatrix} (1+\rho)^2 & 0 \\ \kappa(1+\rho)+\delta\kappa & \delta^2 \end{pmatrix}\end{aligned}$$

The response pattern does not change if we consider orthogonal impulse responses, because the shocks in the model are contemporaneously uncorrelated ($\Sigma_{\mathbf{u}}$ is diagonal).

(e) (i)

$$\begin{aligned}\mathbf{y}_{T+1|T} &= \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}1+\rho & 0\\ \kappa & \delta\end{pmatrix}\begin{pmatrix}0\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ \delta\gamma\end{pmatrix}\\ \mathbf{y}_{T+2|T} &= \begin{pmatrix}1+\rho & 0\\ \kappa & \delta\end{pmatrix}\begin{pmatrix}0\\ \delta\gamma\end{pmatrix} = \begin{pmatrix}0\\ \delta^2\gamma\end{pmatrix}\end{aligned}$$

(ii)

$$\begin{aligned}\Sigma_{\mathbf{y}}(2) &= \sum_{i=0}^{2-1}\Phi_i\Sigma_{\mathbf{u}}\Phi_i'\\ &= \Sigma_{\mathbf{u}} + \mathbf{A}_1\Sigma_{\mathbf{u}}\mathbf{A}_1'\\ &= \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix} + \begin{pmatrix}1+\rho & 0\\ \kappa & \delta\end{pmatrix}\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}1+\rho & \kappa\\ 0 & \delta^2\end{pmatrix}\\ &= \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix} + \begin{pmatrix}(1+\rho)^2\sigma_1^2 & (1+\rho)\kappa\sigma_1^2\\ (1+\rho)\kappa\sigma_1^2 & \kappa^2\sigma_1^2+\delta^2\sigma_2^2\end{pmatrix}\end{aligned}$$

Therefore,

$$\sigma^2_{unemployment}(2) = \kappa^2\sigma_1^2 + (1+\delta^2)\sigma_2^2.$$

(iii) Forecast interval:

$$\mathbf{y}_T \pm z_{0.975}\cdot\sigma^2_{unemployment}$$

where $z_{0.975}$ is the 0.975-quantile from the standard normal distribution. Plugging in, we get

$$\delta^2\gamma \pm 1.96\cdot\sqrt{\kappa^2\sigma_1^2 + (1+\delta)\sigma_2^2}$$

Solution to problem 2

(a)

$$\begin{aligned}\Delta\mathbf{y}_t &= \mathbf{y}_t - \mathbf{y}_{t-1} = (\mathbf{A}_1 - \mathrm{I}_K)\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \mathbf{u}_t\\ &= -(\mathrm{I}_K - \mathbf{A}_1)\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-1} - \mathbf{A}_2\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \mathbf{u}_t\\ &= \underbrace{-(\mathrm{I}_K - \mathbf{A}_1 - \mathbf{A}_2)}_{\Pi}\mathbf{y}_{t-1} \underbrace{-\mathbf{A}_2}_{\Gamma_1}\underbrace{(\mathbf{y}_{t-1} - \mathbf{y}_{t-2})}_{\Delta\mathbf{y}_{t-1}} + \mathbf{u}_t\\ &= \Pi\mathbf{y}_{t-1} + \Gamma_1\Delta\mathbf{y}_{t-1} + \mathbf{u}_t\end{aligned}$$

(b) $\mathrm{rank}(\Pi) = 0$ implies $\Pi = 0$, as the zero matrix is the only matrix of rank zero. Therefore, using the matrices defined in the problem, we get

$$\Delta\mathbf{Y} = \Gamma_1\Delta\mathbf{X} + \mathbf{U}$$

(c)

$$\begin{aligned}\mathrm{vec}(\Delta\mathbf{Y}) &= \mathrm{vec}(\Gamma_1\Delta\mathbf{X}) + \mathrm{vec}(\mathbf{U})\\ &= (\Delta\mathbf{X}'\otimes\mathrm{I}_K)\mathrm{vec}(\Gamma_1) + \mathrm{vec}(\mathbf{U})\end{aligned}$$

Use OLS formula "$\widehat{\beta} = (X'X)^{-1}X'y$" with $X = (\Delta\mathbf{X}' \otimes \mathrm{I}_K)$ $\beta = \text{vec}(\Gamma_1)$ and $y = \text{vec}(\Delta\mathbf{Y})$, and re-arrange using the Kronecker and vec calculation rules:

$$\begin{aligned}
\text{vec}(\widehat{\Gamma}_1) &= \left((\Delta\mathbf{X}' \otimes \mathrm{I}_T)'(\Delta\mathbf{X}' \otimes \mathrm{I}_T)\right)^{-1}(\Delta\mathbf{X}' \otimes \mathrm{I}_T)'\text{vec}(\Delta\mathbf{Y}) \\
&= (\Delta\mathbf{X}\Delta\mathbf{X}' \otimes \mathrm{I}_T)^{-1}(\Delta\mathbf{X} \otimes \mathrm{I}_T)\text{vec}(\Delta\mathbf{Y}) \\
&= \left[(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1}\Delta\mathbf{X} \otimes \mathrm{I}_T\right]\text{vec}(\Delta\mathbf{Y}) \\
&= \text{vec}\left(\Delta\mathbf{Y}\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1}\right).
\end{aligned}$$

Dropping the vec operator gives

$$\widehat{\Gamma}_1 = \Delta\mathbf{Y}\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1}$$

Or, simpler, start with the standard multivariate OLS equation $\mathbf{Y} = \mathbf{X}B + \mathbf{U}$, then the estimator is

$$\hat{B} = \mathbf{Y}\mathbf{X}'(\mathbf{X}\mathbf{X}')^{-1}.$$

Recognising that the equation $\Delta\mathbf{Y} = \Gamma_1\Delta\mathbf{X} + \mathbf{U}$ is of the same format, one can immediately find the estimator in this case.

(d) Reformulate the estimator in terms of the true parameter:

$$\begin{aligned}
\widehat{\Gamma}_1 &= \Delta\mathbf{Y}\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1} \\
&= (\Gamma_1\Delta\mathbf{X} + \mathbf{U})\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1} \\
&= \Gamma_1 + \mathbf{U}\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1}
\end{aligned}$$

Showing consistency means showing that the difference between the true parameter and its estimator converges to zero in probability, as $T \to \infty$. Let $\operatorname*{plim}_{T\to\infty}$ denote the operator for convergence in probability, as $T \to \infty$.

$$\begin{aligned}
\operatorname*{plim}_{T\to\infty}(\widehat{\Gamma}_1 - \Gamma_1) &= \operatorname*{plim}_{T\to\infty}\left(\mathbf{U}\Delta\mathbf{X}'(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1}\right) \\
&= \frac{T}{T}\operatorname*{plim}_{T\to\infty}(\mathbf{U}\Delta\mathbf{X}') \cdot \operatorname*{plim}_{T\to\infty}(\Delta\mathbf{X}\Delta\mathbf{X}')^{-1} \\
&= \underbrace{\operatorname*{plim}_{T\to\infty}\left(\frac{\mathbf{U}\Delta\mathbf{X}'}{T}\right)}_{(*)}\underbrace{\operatorname*{plim}_{T\to\infty}\left(\frac{\Delta\mathbf{X}\Delta\mathbf{X}'}{T}\right)^{-1}}_{=\Phi^{-1},\text{ by WLLN}}
\end{aligned}$$

To show that $(*)$ converges to zero in probability, show convergence in mean square and use that mean square convergence implies convergence in probability.

$$\frac{\mathbf{U}\Delta\mathbf{X}'}{T} \overset{m.s.}{\to} 0 \quad \text{if and only if} \quad \mathbb{E}\left[\frac{\mathbf{U}\Delta\mathbf{X}'}{T}\right] \to 0 \text{ and } \mathrm{V}\left[\frac{\mathbf{U}\Delta\mathbf{X}'}{T}\right] \to 0 \text{ as } T \to \infty$$

We use the vectorized form of $(*)$: Mean:

$$\mathbb{E}\left[\frac{\mathbf{U}\Delta\mathbf{X}'}{T}\right] = \frac{1}{\sqrt{T}}\mathbb{E}\left[\frac{1}{\sqrt{T}}\text{vec}(\mathbf{U}\Delta\mathbf{X}')\right] = 0, \quad \forall t, \tag{1}$$

according to (CLT).
Variance:

$$\mathrm{V}\left[\frac{\mathbf{U}\Delta\mathbf{X}'}{T}\right] = \frac{1}{T}\mathrm{V}\left[\frac{1}{\sqrt{T}}\mathrm{vec}(\mathbf{U}\Delta\mathbf{X}')\right]$$

As $T \to \infty$, $\mathrm{V}\left[\frac{1}{\sqrt{T}}\mathrm{vec}(\mathbf{U}\Delta\mathbf{X}')\right] \to \Phi^{-1}\otimes\Sigma_{\mathbf{u}}$, according to (CLT). Therefore $\frac{1}{T}\mathrm{V}\left[\frac{1}{\sqrt{T}}\mathrm{vec}(\mathbf{U}\Delta\mathbf{X}')\right] \to 0$, i.e. the variance vanishes.
Since we have shown that the estimator converges in mean square to the true parameter, we may conclude that is also converges in probability, i.e., that it is consistent.

Solution to problem 3

(a) Least squares objective function: minimize

$$V(\Lambda, F_t) = \frac{1}{T}\sum_{t=1}^{T}(X_t - \Lambda F_t)'(X_t - \Lambda F_t) \tag{2}$$

with respect to F_t and Λ.

(b) In (2), both Λ and F_t are unknown, and can therefore not be identified without further restrictions. An example restriction is $\Lambda'\Lambda = \mathrm{I}_r$. The number of linear restrictions needed for identification is r^2, or, if you notice that by definition this restriction is symmetric, the number of restrictions is $\frac{1}{2}r(r+1)$.

Solution to problem 4

(a) (i) Covariance matrix of disturbances:

$$\epsilon \sim (0, \Omega)\ , \quad \text{where}$$

$$\begin{aligned}\Omega &:= \mathrm{V}\,(\epsilon) = \underbrace{\mathrm{V}(G\mu)}_{=G\sigma_\mu^2\mathrm{I}_N G'} + \mathrm{V}(e)\\ &= \sigma_\mu^2 GG' + \sigma_e^2\mathrm{I}_{NT}\\ &= \sigma_\mu^2(\mathrm{I}_N\otimes J_T) + \sigma_e^2(\mathrm{I}_N\otimes\mathrm{I}_T)\end{aligned}$$

Note that $\mathrm{I}_N\otimes J_T = T\cdot P_G$; write $\mathrm{I}_T = \frac{1}{T}J_T + (\mathrm{I}_T - \frac{1}{T}J_T)$

$$\begin{aligned}\Omega &= T\sigma_\mu^2 P_G + \sigma_e^2[\mathrm{I}_N\otimes(\frac{1}{T}J_T + (\mathrm{I}_T - \frac{1}{T}J_T))]\\ &= T\sigma_\mu^2 P_G + \sigma_e^2[\underbrace{\mathrm{I}_N\otimes\frac{1}{T}J_T}_{=P_G} + \underbrace{\mathrm{I}_N\otimes(\mathrm{I}_T - \frac{1}{T}J_T)}_{=M_G}]\\ &= \underbrace{(T\sigma_\mu^2 + \sigma_e^2)}_{=:\sigma_1^2} P_G + \sigma_e^2 M_G = \sigma_1^2 P_G + \sigma_e^2 M_G\ ,\end{aligned} \tag{3}$$

(ii) P_G and M_G are symmetric and idempotent, and σ_e^2 and σ_μ^2 are the unique eigenvalues of Ω. Writing the covariance in this form allows us to invert it easily.

(b) Premultiplying the model with $M_0 = I - P_0$, where $P_0 = \mathbb{1}_{NT}(\mathbb{1}'_{NT}\mathbb{1}_{NT})^{-1}\mathbb{1}'_{NT}$ removes the intercept from the model.

$$M_0 y = \alpha \underbrace{M_0 \mathbb{1}_{NT}}_{=0} + M_0 X\beta + M_0 e \tag{4}$$

Running OLS on the transformed data gives an efficient estimator of β:

$$\widehat{\beta} = (X' M_0 X)^{-1} X' M_0 y \tag{5}$$