Ectr. III Example Exam 1

Problem 1 (35/90 points)

Consider the following VAR(2) model for the annual consumer price inflation rate (y_1) and the annual producer price inflation rate (y_2):

$$\mathbf{y}_t = \begin{pmatrix} 0.1 \\ 0.15 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix} \mathbf{y}_{t-1} + \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} \mathbf{y}_{t-2} + \mathbf{u}_t, \quad t = 0, \pm 1, \pm 2, ... \tag{1}$$

where $\mathbf{y}_t = (y_{t1}, y_{t2})'$. The vector of disturbances $\mathbf{u}_t = (u_{t1}, u_{t2})'$ is assumed to be generated by a Gaussian white noise process with fixed covariance matrix

$$\Sigma_{\mathbf{u}} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

(a) (5 points) Write model (1) as VAR(1) model, including the distribution of the vector of disturbances. How can you check whether the process is stable? (Assume in the following that the condition is fulfilled.)

(b) (5 points) Calculate the unconditional mean of the process (1).

(c) (4 points) Does consumer price inflation Granger-cause producer price inflation? Does producer price inflation Granger-cause consumer price inflation? Justify briefly.

(d) (3 points) Compute the impulse responses Φ_h of the VAR given in (1) for horizons $h = 0, 1$. Hint: The impulse responses correspond to the moving average coefficient matrices.

(e) (12 points) Compute the orthogonal impulse responses Θ_h of the VAR in (1) for horizons $h = 0, 1$. Proceed as follows:

(i) Perform a Cholesky decomposition of the error covariance matrix, i.e., find a lower triangular matrix L, for which it holds that

$$\Sigma_{\mathbf{u}} = LL'. \tag{2}$$

Hint: In order to get the elements of L, consider the system of equations implied by $\Sigma_{\mathbf{u}} = LL'$, where the upper right element in L equals 0.

(ii) Compute the orthogonal impulse responses using L and the impulse responses you computed in (d). If you were not able to compute these, give the general formula.

(iii) Explain briefly how the interpretation differs between "ordinary" and orthogonal impulse responses.

(f) (6 points) Suppose that at time $T-1$ the observed values $\mathbf{y}_{T-1} = (y_{T-1,1}, y_{T-1,2})'$ are $(0.11, 0.2)'$ and at time T, the observed values $\mathbf{y}_T = (y_{T1}, y_{T2})$ are $(0.04, 0.1)'$.

(i) Compute the 1-step ahead prediction $\mathbf{y}_T(1)$.

(ii) Compute the forecast error variance for the 2-step ahead prediction of the consumer price inflation rate.

Problem 2 (25/90 points)

Consider a VECM(2) model for a K-dimensional vector of variables $\mathbf{y}_t = (y_{t1}, ..., y_{tK})'$ that are all integrated of order one:

$$\Delta\mathbf{y}_t = \mathbf{c} + \Pi\mathbf{y}_{t-1} + \Gamma_1\Delta\mathbf{y}_{t-1} + \Gamma_2\Delta\mathbf{y}_{t-2} + \mathbf{u}_t, \quad t = 0, \pm1, \pm2, ... \tag{3}$$

where Γ_1 and Γ_2 are $(K \times K)$-coefficient matrices, and $\mathbf{u}_t$ is a K-vector of Gaussian white noise disturbances with zero mean and fixed, nonsingular covariance matrix $\Sigma_{\mathbf{u}}$.

(a) (8 points) Show that the process given in (3) can be written in the form

$$\mathbf{y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \mathbf{A}_3\mathbf{y}_{t-3} + \mathbf{u}_t, \tag{4}$$

including explicit expressions for the coefficient matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ in terms of Π, Γ_1 and Γ_2 from (3).

(b) (6 points) Suppose T observations plus a suitable number of pre-sample observations of $\mathbf{y}_t$ are available. Write the vector error correction model (3) jointly for all observations in matrix notation. Define all model components, including the matrix dimensions. How many pre-sample observations do you need?

(c) (4 points) Using the model in matrix notation from part (b), give the formula for the least squares estimator of the composite coefficient matrix $(\mathbf{c} : \Pi : \Gamma_1 : \Gamma_2)$.

(d) (7 points) Suppose a test has indicated that the variables in $\mathbf{y}_t = (y_{t1}, ..., y_{tK})'$ are not cointegrated. Simplify model (3) such that this information is incorporated. State the assumption you need to prove consistency of the least squares estimator of the unknown coefficients in the simplified model. *Hint: You need a weak law of large numbers for the regressors.*

Problem 3 (10/90 points)

Consider a dynamic factor model for a $(K \times 1)$-vector of time series Y_t, given by the set of equations

$$\begin{aligned} Y_t &= \Lambda_1 f_t + \Lambda_2 f_{t-1} + \epsilon_t, \qquad \epsilon_t \sim i.i.d.N(0, \Sigma_\epsilon), & (5) \\ f_t &= \varphi f_{t-1} + u_t, \qquad u_t \sim i.i.d.N(0, \sigma_u^2), & (6) \end{aligned}$$

where f_t is an unobserved scalar factor, Λ_1 and Λ_2 are $(K \times 1)$- coefficient vectors, and φ and σ_u^2 are scalar coefficients. Σ_ϵ is a $(K \times K)$-diagonal matrix with full rank.

(a) (3 points) Re-write equation (5) in the form of a static factor model, i.e.,

$$Y_t = \mathbf{\Lambda} F_t + \epsilon_t.$$

Define all model components appropriately.

(b) (3 points) Re-write equation (6) in the form

$$F_t = \boldsymbol{\psi} F_{t-1} + G u_t, \qquad (7)$$

with F_t from (a), where G is a deterministic matrix that makes sure (6) and (7) are equivalent. Define all model components appropriately, including G.

(c) (4 points) In practice, the number of factors underlying a data set is unknown. How would you choose this number?

Problem 4 (20/90 points)

Consider the one-way error component panel data model for a sample on N individuals at T time points in matrix notation:

$$y = X\beta + G\mu + e \quad e \sim N(0, \sigma_e^2 \mathrm{I}_{NT}) \tag{8}$$

where

- $\beta = (\beta_1, ..., \beta_K)'$ is a vector of unknown fixed coefficients,
- μ is a $(N \times 1)$-vector of unknown fixed individual-specific intercepts,
- X is a $(NT \times K)$-matrix of nonrandom regressors with rank$(X) = K$,
- y and e with are $(NT \times 1)$-vectors of dependent variable observations and disturbances, respectively,
- $G = \mathrm{I}_N \otimes \mathbb{1}_T$ is a matrix of individual dummies. $\mathbb{1}_T$ denotes a $(T \times 1)$-vector of ones.

(a) (4 points) Define the matrix $P := G(G'G)^{-1}G'$. Transform model (8) by pre-multiplying both sides of the equation by P. What can you say about the structure of the transformed data vector Py and the transformed regressor matrix PX?

(b) (10 points) Define the matrix $Q := \mathrm{I}_{NT} - P$, where P is given in part (a). Derive the expectation and the covariance matrix of the within estimator

$$\widehat{\beta}_W = (X'QX)^{-1}X'Qy. \tag{9}$$

(c) (6 points) Suppose a test has indicated that there is no unobserved heterogeneity in the model, i.e. $\mu = 0$. Suggest an *efficient* estimator for β in (8) under this restriction. Explain briefly why it is not necessary to use the within estimator (9) in this setting.

Appendix

Rules for Kronecker product and vectorization (vec) operator
Assume that the matrices A, B, C, and I (identity matrix) have dimensions such that all products exist.

(K1) $(A \otimes B)' = A' \otimes B'$

(K2) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(K3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

(K4) $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$

(K5) $\text{vec}(AB) = (\mathrm{I} \otimes A)\text{vec}(B) = (B' \otimes \mathrm{I})\text{vec}(A)$

(K6) $\text{vec}(B')'\text{vec}(A) = \text{vec}(A')'\text{vec}(B) = \text{tr}(BA) = \text{tr}(AB)$

(K7) $\text{tr}(ABC) = \text{vec}(A')'(\mathrm{I} \otimes B)\text{vec}(C)$