Ectr. III Example Exam 1: Solution

Solution to problem 1

(a) Companion form:

$$\mathbf{Y}_t = \begin{pmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.15 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.1 & 0 & 0 \\ 0 & 0.4 & 0 & 0.1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \mathbf{Y}_{t-1} + \begin{pmatrix} \mathbf{u}_t \\ 0 \end{pmatrix} \tag{1}$$

with $\begin{pmatrix} \mathbf{u}_t \\ 0 \end{pmatrix} \sim N \begin{pmatrix} \Sigma_\mathbf{u} & 0 \\ 0 & 0 \end{pmatrix}$.

To check stability, check whether all eigenvalues of **A** are smaller than 1 in modulus.

(b) Unconditional mean:

$$\begin{aligned} \mathbb{E}[\mathbf{y}_t] &= (\mathrm{I}_2 - \mathbf{A}_1 - \mathbf{A}_2)^{-1} \cdot \mathbf{c} & (2) \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0.1 \\ 0.15 \end{pmatrix} & (3) \\ &= \frac{1}{0.25} \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 0.26 \\ 0.3 \end{pmatrix} & (4) \end{aligned}$$

(c) Consumer price inflation (y_1) does not Granger-cause producer price inflation (y_2), because $a_{21,1} = a_{21,2} = 0$.

Producer price inflation (y_2) Granger-causes consumer price inflation (y_1), because $a_{12,1} = 0.1 \neq 0$.

(d) $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Phi_1 = A_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix}$.

(e) (i)

$$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} = \begin{pmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{pmatrix} \begin{pmatrix} \ell_{11} & \ell_{21} \\ 0 & \ell_{22} \end{pmatrix} \tag{5}$$

Solving the system of equations:

$$\begin{aligned} \ell_{11}^2 = 1 \quad &\Leftrightarrow \quad \ell_{11} = 1 & (6) \\ \ell_{11} \cdot \ell_{21} = 0.1 \quad &\Leftrightarrow \quad \ell_{21} = 0.1 & (7) \\ \ell_{21}^2 + \ell_{22}^2 = 1 \quad &\Leftrightarrow \quad \ell_{22} = \sqrt{1 - 0.1^2} \approx 0.995 & (8) \end{aligned}$$

(ii) $\Theta_0 = \Phi_0 L = L$ and
$\Theta_1 = \Phi_1 L = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0.1 & 0.995 \end{pmatrix} = \begin{pmatrix} 0.51 & 0.0995 \\ 0.04 & 0.3980 \end{pmatrix}$

(iii) "Ordinary" impulse responses contain the impacts of unit shocks to the variables in the system. Orthogonal impulse responses contain the impacts of shocks to a transformed model in which the shocks have are contemporaneously uncorrelated.

(f) (i)

$$\mathbf{y}_T(2) = \begin{pmatrix} 0.1 \\ 0.15 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} 0.04 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} \begin{pmatrix} 0.11 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.13 \\ 0.21 \end{pmatrix} \tag{9}$$

(ii)

$$MSE[\mathbf{y}_T(2)] = \sum_{i=0}^{1} \Phi_i \Sigma_\mathbf{u} \Phi_i' = \Sigma_\mathbf{u} + \Phi_1 \Sigma_\mathbf{u} \Phi_1' = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix} + \begin{pmatrix} 1.31 & 0.64 \\ 0.64 & 1.16 \end{pmatrix} = \begin{pmatrix} 2.31 & 0.74 \\ 0.74 & 2.16 \end{pmatrix} \tag{10}$$

Forecast error variance of the 2-step ahead prediction of consumer price inflation is 2.31.

Solution to problem 2

(a)

$$\begin{aligned} \mathbf{y}_t - \mathbf{y}_{t-1} &= \mathbf{c} + \Pi \mathbf{y}_{t-1} + \Gamma_1(\mathbf{y}_{t-1} - \mathbf{y}_{t-2}) + \Gamma_2(\mathbf{y}_{t-2} - \mathbf{y}_{t-3}) + \mathbf{u}_t & (11) \\ \mathbf{y}_t &= \mathbf{c} + \underbrace{(\mathrm{I}_K + \Pi + \Gamma_1)}_{\mathbf{A}_1} \mathbf{y}_{t-1} + \underbrace{(\Gamma_2 - \Gamma_1)}_{\mathbf{A}_2} \mathbf{y}_{t-2} + \underbrace{(-\Gamma_2)}_{\mathbf{A}_3} \mathbf{y}_{t-3} + \mathbf{u}_t & (12) \end{aligned}$$

(b)

$$\Delta \mathbf{Y} = \mathbf{B}\mathbf{X} + \mathbf{U} \tag{13}$$

where

$$\mathbf{Y}_{(K\times T)} = (\mathbf{y}_1, ..., \mathbf{y}_T), \quad \mathbf{X}_t = (1, \mathbf{y}_1', \Delta\mathbf{y}_t', \Delta\mathbf{y}_{t-1}')', \quad \mathbf{X}_{((1+3K)\times T)} = (\mathbf{X}_0, ..., \mathbf{X}_{T-1}),$$
$$\mathbf{B}_{(K\times(1+3K))} = (\mathbf{c} : \Pi : \Gamma_1 : \Gamma_2), \quad \mathbf{U}_{(K\times T)} = (\mathbf{u}_1, ..., \mathbf{u}_T)$$

Three pre-sample observations are needed: $\mathbf{y}_0$, $\mathbf{y}_{-1}$, and $\mathbf{y}_{-2}$.

(c) $\widehat{B} = \Delta\mathbf{Y}\mathbf{X}'(\mathbf{X}\mathbf{X}')^{-1}$

(d) (3 points) Simplified model:

$$\Delta\mathbf{y}_t = c + \Gamma_1 \Delta\mathbf{y}_{t-1} + \Gamma_2 \Delta\mathbf{y}_{t-2} + \mathbf{u}_t \tag{14}$$

(4 points) Needed: WLLN for regressors $\Delta\mathbf{X} = (\Delta\mathbf{X}_0, ..., \Delta\mathbf{X}_{T-1})$ with $\Delta\mathbf{X}_t = (1, \Delta\mathbf{y}_t', \Delta\mathbf{y}_{t-1}')'$:

$$\lim_{T\to\infty} \frac{1}{T} \Delta\mathbf{X}\Delta\mathbf{X}' = \Omega, \tag{15}$$

where Ω is fixed finite and positive definite matrix.

Solution to problem 3

(a)

$$\begin{aligned} X_t &= \mathbf{\Lambda} F_t + \epsilon_t \\ &= \underbrace{(\Lambda_1 \quad \Lambda_2)}_{\mathbf{\Lambda}} \underbrace{\begin{pmatrix} f_t \\ f_{t-1} \end{pmatrix}}_{F_t} + \epsilon_t, \end{aligned}$$

with $\epsilon_t \sim i.i.d.N(0, \Sigma_\epsilon)$

(b)

$$\begin{aligned} F_t = \begin{pmatrix} f_t \\ f_{t-1} \end{pmatrix} &= \boldsymbol{\psi} F_{t-1} + G u_t \\ &= \begin{pmatrix} \psi & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{t-1} \\ f_{t-2} \end{pmatrix} + \begin{pmatrix} u_t \\ 0 \end{pmatrix} \end{aligned}$$

so that

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$G u_t \sim i.i.d.N(0, \underbrace{\sigma_u^2 G G'}_{\begin{pmatrix} \sigma_u^2 & 0 \\ 0 & 0 \end{pmatrix}})$$

(c) One can use information criteria. The number of factors r is chosen such that it minimizes the objective function (the log likelihood) plus a penalty term that increases in r, with respect to r.

Solution to problem 4

(a) (1 point) Transformed model:

$$Py = PX\beta + PG\mu + Pe$$

(2 points) Py consists of vectors of individual specific means over time:

$$Py = \begin{pmatrix} \mathbb{1}_T \cdot \sum_{t=1}^{T} y_{1t} \\ \vdots \\ \mathbb{1}_T \cdot \sum_{t=1}^{T} y_{Nt} \end{pmatrix} \tag{16}$$

(1 point) The columns of the matrix PX contain the individual-specific means of the regressor observations over time.

(b) (5 points) Expectation:

$$\begin{aligned}
\mathbb{E}[\widehat{\beta}_W] &= \mathbb{E}[(X'QX)^{-1}X'Qy] && (17)\\
&= (X'QX)^{-1}X'Q\mathbb{E}[X\beta + G\mu + e] && (18)\\
&= (X'QX)^{-1}X'Q(QX\beta + \underbrace{QG}_{=0}\mu + \underbrace{\mathbb{E}[Qe]}_{=0}) && (19)\\
&= (X'QX)^{-1}X'QX\beta && (20)\\
&= \beta && (21)
\end{aligned}$$

(5 points) Covariance matrix:

$$\begin{aligned}
\mathrm{V}[\widehat{\beta}_W] &= \mathrm{V}[\beta + (X'QX)^{-1}X'Qe] && (22)\\
&= (X'QX)^{-1}X'Q \underbrace{\mathrm{V}[e]}_{=\sigma_e^2 \mathrm{I}_{NT}} QX(X'QX)^{-1} && (23)\\
&= \sigma_e^2 (X'QX)^{-1} && (24)
\end{aligned}$$

Throughout, we have used the projection properties of Q: $Q = Q' = QQ$.

(c) (3 points) In a model without individual heterogeneity, the pooled OLS estimator is efficient:

$$\widehat{\beta} = (X'X)^{-1}X'y. \tag{25}$$

(3 points) Using $\widehat{\beta}_W$ in this setting implies subtracting individual-specific means from each data point. This is not necessary here.