

Ectr. III Example Exam 2: Solution

Solution to problem 1

- (a) To check stability, check whether all roots of the reverse characteristic polynomial are larger than 1 in modulus.

$$\det(I - Az) = \det \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.8z & 0.2z \\ 0 & 0.5z \end{pmatrix} \right) = (1 - 0.8z)(1 - 0.5z) \quad (1)$$

Therefore $1 - 0.8z = 0 \Leftrightarrow z = 1.25$ or $1 - 0.5z = 0 \Leftrightarrow z = 2$.

- (b) Unconditional mean:

$$\mathbb{E}[\mathbf{y}_t] = \left(\mathbf{I}_2 - \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} \right)^{-1} \cdot \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} \quad (2)$$

$$= \frac{1}{0.1} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 0.2 \end{pmatrix} \quad (3)$$

- (c) Profit growth (y_1) does not Granger-cause investment growth (y_2), because $a_{21,1} = 0$.
Investment growth (y_2) Granger-causes profit growth (y_1), because $a_{12,1} = 0.2 \neq 0$.

- (d) $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Phi_1 = \mathbf{A}_1 = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix}$ and

$$\Phi_2 = \mathbf{A}_2 = \mathbf{A}_1^2 = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.64 & 0.26 \\ 0 & 0.25 \end{pmatrix}$$

A unit shock to profit growth has no impact on investment growth for any h . Reason: no Granger causality; impulse responses are zero.

- (e) No, because the shocks are not contemporaneously correlated.

- (f) (i)

$$\mathbf{y}_T(1) = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.52 \\ 0.1 \end{pmatrix} \quad (4)$$

$$\mathbf{y}_T(2) = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.52 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.636 \\ 0.15 \end{pmatrix} \quad (5)$$

- (ii)

$$MSE[\mathbf{y}_T(2)] = \sum_{i=0}^1 \Phi_i \Sigma_{\mathbf{u}} \Phi_i' = \Sigma_{\mathbf{u}} + \Phi_1 \Sigma_{\mathbf{u}} \Phi_1' = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix} + \begin{pmatrix} 0.216 & 0.06 \\ 0.06 & 0.15 \end{pmatrix} = \begin{pmatrix} 0.516 & 0.06 \\ 0.06 & 0.75 \end{pmatrix} \quad (6)$$

- (iii) 95% prediction interval:

$$\hat{y}_{1,T}(2) \pm 1.96 \cdot \sqrt{MSE[\mathbf{y}_T(2)]_{11}} = 0.636 \pm 1.96 \cdot \sqrt{0.216} \approx [-0.275, 1.547]$$

Solution to problem 2

(a) (i) Model:

$$\Delta \mathbf{y}_t = \alpha \beta' \tilde{\mathbf{y}}_{t-1} + \mathbf{u}_t, \quad (7)$$

with

$$\Delta \mathbf{y}_t = \begin{pmatrix} \Delta y_{1t} \\ \Delta y_{2t} \\ \Delta y_{3t} \end{pmatrix} \quad (8)$$

where \mathbf{y}_t , \mathbf{y}_{t-1} and \mathbf{u}_t are (3×1) -vectors and α and β are (3×2) -matrices. According to the economic theory, we have

$$\beta = \begin{pmatrix} 1 & 1 \\ -0.3 & 0 \\ 0 & -2 \end{pmatrix} \quad (9)$$

(ii) New model:

$$\Delta \tilde{\mathbf{y}}_t = \alpha \beta' \tilde{\mathbf{y}}_{t-1} + \mathbf{u}_t, \quad (10)$$

with

$$\Delta \tilde{\mathbf{y}}_t = \begin{pmatrix} \Delta y_{2t} \\ \Delta y_{3t} \\ \Delta y_{1t} \end{pmatrix}, \quad (11)$$

and

$$\beta = \begin{pmatrix} -0.3 & 0 \\ 0 & -2 \\ 1 & 1 \end{pmatrix} \Leftrightarrow \tilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{0.3} & -\frac{1}{2} \end{pmatrix} \quad (12)$$

(b) Model in matrix notation:

$$\Delta \mathbf{Y} = \Pi \mathbf{Y}_{-1} + \mathbf{U} = \alpha \beta' \mathbf{Y}_{-1} + \mathbf{U} \quad (13)$$

Vectorized model:

$$\text{vec}(\Delta \mathbf{Y}) = (\mathbf{Y}_{-1}' \beta \otimes \mathbf{I}_K) \text{vec}(\alpha) + \text{vec}(\mathbf{U}) \quad (14)$$

OLS estimator:

$$\text{vec}(\hat{\alpha}) = [(\mathbf{Y}_{-1}' \beta \otimes \mathbf{I}_K)' (\mathbf{Y}_{-1}' \beta \otimes \mathbf{I}_K)]^{-1} (\mathbf{Y}_{-1}' \beta \otimes \mathbf{I}_K)' \text{vec}(\Delta \mathbf{Y}) \quad (15)$$

$$= (\beta' \mathbf{Y}_{-1}' \mathbf{Y}_{-1}' \beta \otimes \mathbf{I}_K)^{-1} ((\beta' \mathbf{Y}_{-1}' \otimes \mathbf{I}_K) \text{vec}(\Delta \mathbf{Y})) \quad (16)$$

$$= \text{vec}[\Delta \mathbf{Y} \mathbf{Y}_{-1}' \beta (\beta' \mathbf{Y}_{-1}' \mathbf{Y}_{-1}' \beta)^{-1}] \quad (17)$$

(c) Test sequence:

1. $H_0: \text{rank}(\Pi) = 0$ vs. $H_1: \text{rank}(\Pi) > 0$
2. $H_0: \text{rank}(\Pi) = 1$ vs. $H_1: \text{rank}(\Pi) > 1$
3. $H_0: \text{rank}(\Pi) = 2$ vs. $H_1: \text{rank}(\Pi) = 3$

Decision rule: Terminate the test sequence once H_0 is not rejected for the first time.

Solution to problem 3

(a) Λ_1 and Λ_2 are $(K \times 2)$ -matrices, φ is a (2×2) -matrix, Σ_ϵ is a $(K \times K)$ -matrix, and Σ_u is a (2×2) -matrix.

(b)

$$\mathbf{Y}_t = \Lambda \mathbf{F}_t + \epsilon_t \quad (18)$$

with

$$\Lambda_{(K \times 4)} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \end{pmatrix} \quad \text{and} \quad \mathbf{F}_t = \begin{pmatrix} f_t \\ f_{t-1} \end{pmatrix} \quad (19)$$

(c)

$$F_t = \psi F_{t-1} + Gu_t \quad (20)$$

$$\text{with } F_t \text{ as in (b), } \psi_{(4 \times 4)} = \begin{pmatrix} \varphi & 0 \\ I_2 & 0 \end{pmatrix} \text{ and } G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution to problem 4

(a) Transformed model:

$$Qy = QX\beta + \underbrace{QG}_{=0}\mu + Qe$$

Qy is a $(NT \times 1)$ -vector containing the differences between observations y_{it} and the individual-specific means of the dependent variable observations over time. The columns of the matrix QX contain the differences between regressor observations and the individual-specific means of the regressor observations over time.

The OLS estimator/within estimator is a good choice to estimate this model, as it is efficient.

(b) (5 points) Expectation:

$$\mathbb{E}[\hat{\beta}_W | X] = \mathbb{E}[(X'QX)^{-1}X'Qy | X] \quad (21)$$

$$= (X'QX)^{-1}X'Q\mathbb{E}[X\beta + G\mu + e | X] \quad (22)$$

$$= (X'QX)^{-1}X'Q(QX\beta + \underbrace{QG}_{=0}\mu + \underbrace{Q\mathbb{E}[e | X]}_{=0}) \quad (23)$$

$$= (X'QX)^{-1}X'QX\beta \quad (24)$$

$$= \beta \quad (25)$$

(5 points) Covariance matrix:

$$V[\hat{\beta}_W | X] = V[\beta + (X'QX)^{-1}X'Qe] \quad (26)$$

$$= (X'QX)^{-1}X'Q\underbrace{V[e | X]}_{=\sigma_e^2 I_{NT}}QX(X'QX)^{-1} \quad (27)$$

$$= \sigma_e^2 (X'QX)^{-1} \quad (28)$$

Throughout, we have used the projection properties of Q : $Q = Q' = QQ$.

(c) Individual-specific effects capture unobserved time-invariant heterogeneity between units. If units are heterogeneous, including fixed effects allows us to estimate the structural coefficients of interest without omitted variable bias.