Dynamical Systems Exam 2

May 30, 2023, 12:15–14:15

Instructions

- This exam counts for 40% of your course grade.
- This exam consists of three (3) questions. The words "END OF EXAM" appear after the last question.
- Show all work and justifications.

Problems

1. For $x, y \in \mathbb{R}$ and a parameter $\mu \in \mathbb{R}$ consider the dynamical system

$$\dot{x} = \mu y + yx^2,$$
$$\dot{y} = x + y^2.$$

   a) [20%] Fix $\mu = 1$.

      i. Determine all equilibria and determine whether they are hyperbolic or not.

      ii. For each equilibrium, compute its index. What does the index tell you about the possibility of the existence of a periodic orbit that surrounds that equilibrium but no other equilibria?

   b) [10%] Now consider μ as a varying parameter. Confirm that $(x, y) = (0, 0)$ is a nonhyperbolic equilibrium for $\mu = 0$. Does the equilibrium $(x, y) = (0, 0)$ undergo a Hopf bifurcation at $\mu = 0$?

2. For $x \in \mathbb{R}$ consider the one-dimensional system

$$\dot{x} = x^2.$$

   a) [5%] Compute the equilibria of the system and classify them. Describe the dynamics for initial conditions near the equilibria.

   b) [15%] Introduce the homogeneous coordinates

$$x = \frac{y}{z} \qquad\qquad y^2 + z^2 = 1$$

   and write down the differential equation for the compactified system on the Poincaré "sphere" (a circle in this case). Sketch the phase portrait of compactified system on the upper hemisphere. Describe the dynamics at infinity, that is, at $z = 0$. Are there any equilibria? If so, classify the equilibria (as equilibria on the invariant circle).

   c) [10%] For $x \neq 0$ consider the transformed coordinate $\xi = \frac{1}{x}$. Derive the differential equation for ξ, determine the compactified system on the Poincaré sphere, and determine the dynamics at infinity. What does this tell you about the dynamics close to $x = 0$?

3. For parameters $\mu, r \in \mathbb{R}$ consider

$$\dot{x} = x^3 + x\mu^2 - xr.$$

a) [30%] Fix $r = 1$.

   i. Compute the equilibria (depending on μ) and their stability.

   ii. Identify potential bifurcation points and determine the type of bifurcation (if any).

   iii. Sketch the bifurcation diagram in the (μ, x) plane.

b) [10%] Now consider how the bifurcations of equilibria in μ change as r is varied. Determine the bifurcation points of each equilibrium as a function of μ and r, that is, for each equilibrium $x^*(\mu, r)$ determine a curve $\mu(r)$ where the equilibrium undergoes a bifurcation. Plot the curves in the (μ, r) parameter plane and describe qualitatively the dynamics in the regions between the curves.

Good luck!
END OF EXAM