Dynamical Systems Exam 1

Mar 27, 2023; 08:30–10:30

Instructions

- This exam counts for 40% of your course grade.
- This exam consists of three (3) questions. The words "END OF EXAM" appear after the last question.
- Show all work and justifications.

Problems

1. For a parameter $p \in \mathbb{R}$ define the matrix

$$A = \begin{pmatrix} 0 & -p & -p \\ -1 & -1 & -1 \\ 1 & p+1 & p+1 \end{pmatrix}$$

and $\mathbf{x} \in \mathbb{R}^3$ consider the linear dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$.

   a) [10%] Compute the eigenvectors and eigenvalues of A. Is the matrix diagonalizable (and does this property depend on p)?
   b) [10%] Is the equilibrium $\mathbf{x}^* = (0,0,0)$ hyperbolic? Determine the stable subspace as a function of p.
   c) [20%] For $p = 1$ compute the matrix exponential $\exp(tA)$ and determine the solution for the initial condition $\mathbf{x}(0) = (1,0,0)$.

2. a) [10%] Suppose that B is a (real) $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. For a Lipschitz continuous function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ consider the inhomogeneous linear dynamical system

$$\dot{\mathbf{x}} = B\mathbf{x} + \mathbf{f}(t).$$

   Prove the variations of constants formula

$$\mathbf{x}(t) = \exp(tB)\mathbf{x}(0) + \int_0^t \exp((t-s)B)\mathbf{f}(s)\mathrm{d}s.$$

   Hint: Consider the time derivative of $\exp(-tB)\mathbf{x}(t)$.

   b) [20%] Now consider consider the differential equation

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y - x^2 \end{aligned}$$

      i. Find an explicit solution for the above system with initial values $x(0) = x_0$ and $y(0) = y_0$.
      ii. Use the answer in (i.) to find explicit formulas for the local stable and unstable manifolds at the equilibrium point $(0,0)$.
      iii. Sketch the phase plane and indicate some solution curves.

3. For $x, y \in \mathbb{R}$ consider the dynamical system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2) \\ \dot{y} &= x + y - y(x^2 + y^2). \end{aligned}$$

a) [15%] Linearize the system and classify the equilibrium point $(x, y) = (0, 0)$ at the origin for the linearized system: Specifically, determine whether it is stable or unstable and what the stable and unstable subspaces are. Sketch the phase plane of the linearized system. Does the sketch reflect the behavior of solutions of the nonlinear system close to the origin?

b) [15%] Show that for sufficiently large $\delta > 0$ the ball

$$B_\delta(0, 0) = \{(x, y) \mid x^2 + y^2 < \delta^2\}$$

is forward invariant for the flow ϕ_t generated by the differential equation. (Recall that a set S is forward invariant for ϕ_t if for every $(x, y) \in S$ there exists a time $T(x, y) > 0$ such that $\phi_t(x, y) \in S$ for all $t \in [0, T(x, y)]$.)
Hint: You can show this by finding a $\delta > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) < 0$ at $x^2 + y^2 = \delta^2$ (but you have to justify why this is sufficient).

c) [10% Extra Credit] Show that there is an annulus

$$R_{r_1, r_2} = \{(x, y) \mid r_1^2 < x^2 + y^2 < r_2^2\}$$

with $r_1, r_2 > 0$ that is forward invariant. Are there any equilibria or periodic orbits in R_{r_1, r_2}?

Good luck!
END OF EXAM