Dynamical Systems Exam 2

May 31, 2022, 12:15–14:15

Instructions

- This exam counts for 40% of your course grade.
- This exam consists of three (3) questions. The words "END OF EXAM" appear after the last question.
- Show all work and justifications.

Problems

1. For $x, y \in \mathbb{R}$ and a parameter $\mu \in \mathbb{R}$ consider the dynamical system

$$\dot{x} = -2\mu x - xy^2,$$
$$\dot{y} = 3\mu^3 y + yx^2.$$

   a) [10%] Compute the linear stability of the equilibrium $(x, y) = (0, 0)$ and determine for which μ the equilibrium is hyperbolic.

   b) [10%] For $\mu > 0$ use index theory to deduce that there cannot be a periodic orbit in any neighborhood of $(x, y) = (0, 0)$ that contains no other equilibria.

   c) [10%] Now consider the more general system

$$\dot{x} = xf(x, y, \mu)$$
$$\dot{y} = yg(x, y, \mu)$$

   where f, g depend smoothly on the dynamical variables x, y as well as the parameter μ. Can the equilibrium $(x, y) = (0, 0)$ undergo a Hopf bifurcation?

2. Consider the linear system

$$\dot{x} = -\omega y$$
$$\dot{y} = \omega x$$

   where $\omega \in \mathbb{R}$ is a parameter.

   a) [10%] Compute the equilibria of the system and classify them. Sketch the dynamics on the phase plane (x, y).

   b) [10%] Use the homogeneous coordinates

$$x = \frac{\xi_1}{\zeta} \qquad y = \frac{\xi_2}{\zeta} \qquad \xi_1^2 + \xi_2^2 + \zeta^2 = 1$$

   and write down the differential equation for the compactified system on the Poincaré sphere.

   c) [10%] Describe the dynamics at infinity, that is, near $\zeta = 0$. Are there any equilibria? Sketch the dynamics on the phase plane (ξ_1, ξ_2).

   d) [10% extra credit] Solve parts (b) and (c) for the nonlinear system

$$\dot{x} = \mu x - \omega y - x(x^2 + y^2)$$
$$\dot{y} = \omega x + \mu y - y(x^2 + y^2).$$

3. For a parameter $\mu \in \mathbb{R}$ consider

$$\dot{x} = x^2 - x + x\mu^2.$$

a) [10%] Compute the equilibria (depending on μ) and their stability.

b) [10%] Identify potential bifurcation points and determine the type of bifurcation (if any).

c) [10%] Sketch the bifurcation diagram in the (μ, x) plane.

d) [10%] Now consider a more general system $\dot{x} = f(x, \mu)$ where again $\mu \in \mathbb{R}$ is a parameter. Suppose that f satisfies $f(-x, \mu) = -f(x, \mu)$ for all x, μ. Show that $x^* = 0$ is an equilibrium for all μ and that x^* cannot undergo a saddle-node or transcritical bifurcation.

Good luck!
END OF EXAM