Dynamical Systems Exam 1

Mar 28, 2022, 08:30–10:30

Instructions

- This exam counts for 40% of your course grade.
- This exam consists of three (3) questions. The words "END OF EXAM" appear after the last question.
- Show all work and justifications.

Problems

1. For the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

   and $\mathbf{x} \in \mathbb{R}^3$ consider the linear dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$.

   a) [10%] Compute the eigenvectors and eigenvalues of A. Is the matrix diagonalizable?

   b) [10%] Is the equilibrium $\mathbf{x}^* = (0, 0, 0)$ hyperbolic? Calculate the stable subspace.

   c) [15%] Compute the matrix exponential $\exp(tA)$ and determine the solution for the initial condition $\mathbf{x}(0) = (1, 2, 3)$.

2. For $x, y \in \mathbb{R}$ consider the dynamical system

$$\dot{x} = y$$
$$\dot{y} = \cos(x)$$

   a) [10%] Show that the system is Hamiltonian and find a Hamiltonian.

   b) [15%] Compute all equilibrium points and classify as a stable/unstable node, center, etc. Moreover, determine the stable, unstable, and center subspaces for the linearized dynamics and for each equilibrium sketch the linearized dynamics. For which equilibria does the sketch reflect the behavior of solutions of the nonlinear system close to the equilibrium according to the Grobman–Hartman theorem?

   c) [5%] Sketch the phase plane of the (nonlinear) system and compare the behavior of trajectories close to equilibria with the results obtained in b) for the linearized dynamics.

3. Suppose that B is a (real) $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. For clarity, we explicitly write the time dependency $\mathbf{x} = \mathbf{x}(t)$.

   a) [15%] For a Lipschitz continuous function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ consider the inhomogeneous linear dynamical system

   $$\dot{\mathbf{x}}(t) = B\mathbf{x}(t) + \mathbf{f}(t).$$

   Derive the variations of constants formula

   $$\mathbf{x}(t) = \exp(tB)\mathbf{x}(0) + \int_0^t \exp((t-s)B)\mathbf{f}(s)\mathrm{d}s.$$

   Hint: Consider the time derivative of $\exp(-tB)\mathbf{x}(t)$.

   b) [10%] Let K be a real $n \times n$ matrix and for $\tau > 0$ let $\Psi : [-\tau, 0] \to \mathbb{R}^n$ be Lipschitz continuous. Consider the *delay differential equation*

   $$\dot{\mathbf{x}}(t) = B\mathbf{x}(t) + K\mathbf{x}(t - \tau).$$

   Use variations of constants to compute a solution $\mathbf{x} : (-\tau, \tau) \to \mathbb{R}^n$ of the delay differential equation with "initial condition" $\mathbf{x}(t) = \Psi(t)$ for $t \in (-\tau, 0]$.
   Hint: Consider $\mathbf{f}(t) = K\mathbf{x}(t - \tau)$ for $t \in [0, \tau]$.

   c) [10%]

      i. Show that the solution $\mathbf{x}(t)$ can be extended for all times $t > 0$.

      ii. The solution becomes more differentiable as time evolves: Show that $\mathbf{x}(t)$ is k-times differentiable as a function on the time interval $((k-1)\tau, k\tau)$.

Good luck!
END OF EXAM