Name:

Midterm 2 (final) Department of Mathematics
Dynamical Systems 637 College of Science

Date: Tuesday May 25, 2021, 12:15 - 14:15
Instructions: 3 questions.
Please show all work and answers.

Final grade: # ptn/10.

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(1) Consider the 1-dimensional differential equation

$$\dot{x} = x(1-x) - \mu^2.$$

a) [10%] Find the bifurcation points and determine their types;
b) [10%] Determine the equilibrium points as function of the parameter $\mu \in \mathbb{R}$;
c) [10%] Give a sketch of the bifurcation diagram.

(2) Consider the system

$$\dot{x} = x(4 - y - x^2);$$
$$\dot{y} = y(x-1).$$

a) [10%] Find all equilibrium points and determine their behavior;
b) [10%] Show that the x-axis and y-axis are invariant for the flow and draw the flow-lines on the axis;
c) [10%] Use index theory to rule out periodic orbits for the above system.

(3) Consider the following system of differential equations:

$$\dot{x} = 2xy - 4y;$$
$$\dot{y} = -x^2 + 4y^2.$$

A symmetry for a system of differential equations is a transformation of the variables x, y and t which leaves the equations unchanged. For example: for the equation $\dot{x} = -x$ the transformation $t \mapsto -t$, $x \mapsto -x$ and $y \mapsto y$ is such a symmetry. As a consequence solutions inherit this symmetry, i.e. if $x(t)$ is a solution, then also $-x(-t)$ is a solution. This principle can be used to find additional solutions or derive properties above the global behavior.

a) [10%] Show that the transformation $t \mapsto -t$, $x \mapsto x$ and $y \mapsto -y$ is a symmetry for the above system;

In order to understand the asymptotic behavior of the flow we can compactify the system to the Poincaré sphere as described in Perko (Sect. 3.10) and the lecture notes (Sect.'s 9.1 and 9.2).

b) [10%] Use the homogeneous coordinates

$$x = \frac{\xi_1}{\zeta}, \quad y = \frac{\xi_2}{\zeta}, \quad \xi_1^2 + \xi_2^2 + \zeta^2 = 1,$$

and write down the differential equations for the compactified system on the Poincaré sphere;

The equator $\{\zeta = 0\}$ describes the 'behavior at infinity' of the system. By substituting $\zeta = 0$ in the compactified system one can study the 'system at infinity'. The diiferential equation for ζ describes how solutions behave 'near infinity'.

c) [10%] Compute the equilibrium points for the system at infinity, i.e. at $\zeta = 0$;

In order to analyze the nature of the critical points we project the Poincaré sphere onto the $(x, 1, z)$-plane in $\mathbb{R}^3$. The equations for the system coordinates $\bar{x}$ and $\bar{z}$ are:

$$\begin{aligned} \dot{\bar{x}} &= \bar{z}^m f\Big(\frac{\bar{x}}{\bar{z}}, \frac{1}{\bar{z}}\Big) - \bar{z}^m g\Big(\frac{\bar{x}}{\bar{z}}, \frac{1}{\bar{z}}\Big)\bar{x}, \\ \dot{\bar{z}} &= -\bar{z}^{m+1} g\Big(\frac{\bar{x}}{\bar{z}}, \frac{1}{\bar{z}}\Big), \end{aligned}$$

where $f(x, y) = 2xy - 4y$ and $g(x, y) = -x^2 + 4y^2$ and where m is chosen in 3b).

d) [10%] Express the points at infinity found in 3c) in terms of $\bar{x}$ and $\bar{z}$ and determine their nature using the above system of differential equations for $\bar{x}$ and $\bar{z}$.

e) [10%][1] Use 3a) to determine the behavior of the remaining points at infinity that project to the $(x, -1, z)$-plane in $\mathbb{R}^3$ and describe the flow at the 'circle at infinity' for the compactified system in 3b)-c).

Good luck!

[1]Extra credit problem.