Midterm 2 (final), 2021

1. $\dot{x} = x(1-x) - \mu^2$, $f(x,\mu) = x(1-x) - \mu^2$

a) Bif. equ's $\quad f(x,\mu) = 0$
$\quad\quad f_x(x,\mu) = 0$

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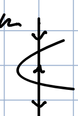
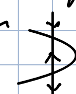
$$\Rightarrow \quad x(1-x) - \mu^2 = 0$$
$$1 - 2x = 0 \Longrightarrow x_* = \frac{1}{2}$$
$$\Rightarrow \quad \frac{1}{2}\left(1-\frac{1}{2}\right) = \mu^2 = \frac{1}{4} \Rightarrow \mu = \pm\frac{1}{2}$$

Candidate bif. pts $(\mu_*, x_*) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Higher derivatives: $f_\mu(x,\mu) = -2\mu$, $f_{xx}(x,\mu) = -2$

For both pts: $f_\mu\left(\frac{1}{2}, \pm\frac{1}{2}\right) = \mp 1$, $f_{xx}\left(\frac{1}{2}, \pm\frac{1}{2}\right) = -2$

Conclusion: $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ are saddle-node bifurcations

normal form $-\mu - x^2$  normal form $\mu - x^2$ 

b) $x(1-x) - \mu^2 = 0 \Leftrightarrow x^2 - x + \mu^2 = 0$

$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \mu^2 = 0$$

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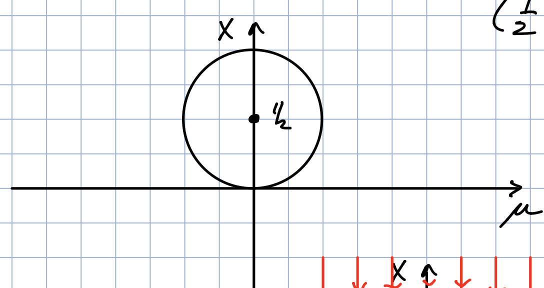
$\Longleftrightarrow \left(x - \frac{1}{2}\right)^2 + \mu^2 = \frac{1}{4}$ circle with center $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.



c)

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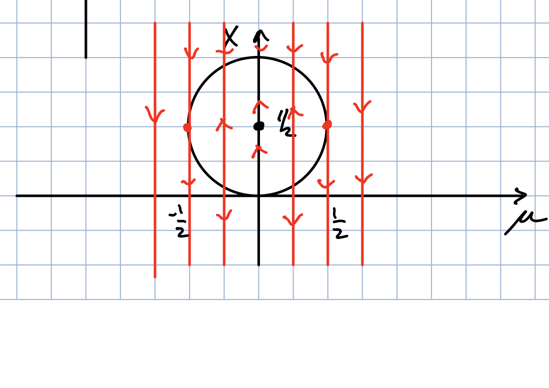

2. a)

$x(4-y-x^2)=0$

$y(x-1)=0 \qquad y=0 \quad \text{or} \quad x=1$

$y=0 \implies x(4-x^2)=0 \implies x=0, \quad x=\pm 2$

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pts. $(0,0), (2,0), (-2,0)$

$x=1 \implies 3-y=0 \implies y=3 \quad$ pt. $(1,3)$.

$$Df = \begin{pmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{pmatrix}$$

$(0,0)$ $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ saddle pt.

$(2,0)$ $\begin{pmatrix} -8 & -2 \\ 0 & 1 \end{pmatrix}$ saddle pt.

$(-2,0)$ $\begin{pmatrix} -8 & 2 \\ 0 & -3 \end{pmatrix}$ stable node

$(1,3)$ $\begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$

$(\lambda+2)\lambda+3=0, \quad \lambda^2+2\lambda+3=0$

$\lambda = \dfrac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2} \implies$ Saddle pt.

b)

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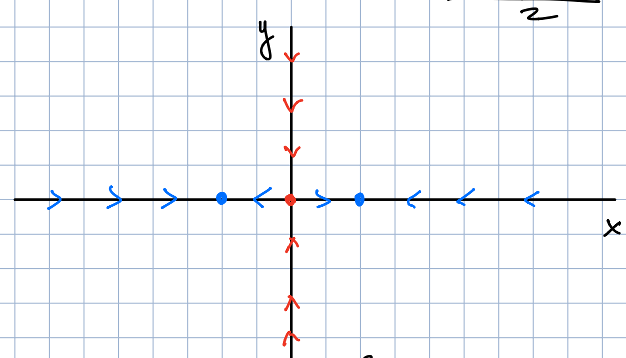
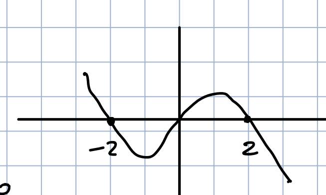


$x=0 \implies \dot{x}=0, \quad \dot{y}=-y$ vector field is tangent to $\{x=0\}$

If $x(0)=0$, then $x(t)=0 \quad \forall t \in \mathbb{R} \implies \{x=0\}$ is invariant.

$y=0 \implies \dot{y}=0, \quad \dot{x}=4x-x^3$

vector field at $\{y=0\}$ is tangent to $\{y=0\}$. If $y(0)=0$, then $y(t)=0 \quad \forall t \in \mathbb{R}$.



c) Fixed pts indices: If (x_*, y_*) is hyperbolic

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we use: saddle pts $i_f(x_* y_*) = -1$

(un)stable node $i_f(x_*, y_*) = +1$

(un)stable focus $i_f(x_*, y_*) = +1$

Index of a periodic orbit γ: $i_f(\gamma) = +1$.

We have the sum formula for indices: if γ is periodic orbit, then

$$1 = i_f(\gamma) = \sum_i i_f(x_i, y_i),$$

where (x_i, y_i) are the fixed pts enclosed by γ.

i) Since the coordinate axis are invariant a periodic orbit cannot intersect them due to the uniqueness of the initial value problem. Therefore a periodic orbit must be in one of the 4 quadrants.

By the index formula we have:

$$+1 = i_f(\gamma) = \sum_i i_f(x_i, y_i)$$

which implies that γ must enclose a fixed pt.

The only fixed pt in the 4 quadrants is $(1,3)$.

We have $i_f(1,3) = +1$, since $(1,3)$ is a stable focus.

It remains to show that there are no periodic orbits enclosing $(1,3)$.

ii) To rule out a periodic orbit we argue as follows.

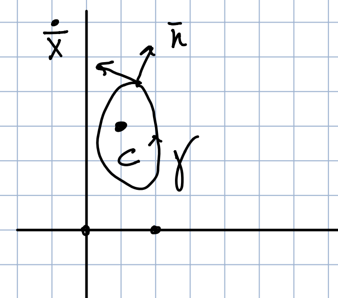
Dulac's criterion in the first quadrant $x>0, y>0$

Seek a function $g(x,y)$ defined on $\{x>0, y>0\}$ (smooth) such that

$$\operatorname{div}\left(g(x,y)\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}\right) \neq 0 \quad \forall\, x>0, y>0.$$

Suppose g exists. Assume γ is a periodic orbit enclosing $(1,3)$ in the first quadrant.

$$\iint_C \operatorname{div}\left(g(x,y)\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}\right) dx\,dy$$



$$\underset{\text{Gauss/Green}}{=} \oint_\gamma g(x,y)\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}\cdot \bar n\, ds$$

$= 0$ since $\dot{\bar x}\cdot \bar n = 0$

On the other hand $\operatorname{div}(g(x,y)\dot{\bar x}) \neq 0$ ↯

We show now that such a function exists.

Try $g(x,y) = \dfrac{1}{xy}$, well-defined for $x>0, y>0$

$$\operatorname{div}\left(g(x,y)\,\dot{\bar x}\right) = \frac{\partial}{\partial x}\left(\frac{4-y-x^2}{y}\right) + \frac{\partial}{\partial y}\left(\frac{x-1}{x}\right)$$

$$= -\frac{2x}{y} < 0$$

Conclusion: There are no periodic orbits.

3.
$$\dot{x} = 2xy - 4y$$
$$\dot{y} = -x^2 + 4y^2$$

a) $t \longmapsto -t, \quad x \longmapsto x, \quad y \longmapsto -y$

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$\Longrightarrow \quad \dot{x} \to -\dot{x}, \quad 2xy - 4y \longrightarrow -2xy + 4y$

$\Longrightarrow$ eqn. 1 remains unchanged.

$\dot{y} \to --\dot{y} = \dot{y}, \quad -x^2+4y^2 \longrightarrow -x^2+4y^2$

$\Longrightarrow$ eqn. 2 remains unchanged.

$\Longrightarrow$ The system of eqn's is invariant under the transformations.

b) $$\bar{f}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2xy - 4y \\ -x^2+4y^2 \end{pmatrix}$$

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$$f^*(\xi_1, \xi_2, \zeta) = \zeta^m \begin{pmatrix} 2\frac{\xi_1}{\zeta}\frac{\xi_2}{\zeta} - 4\frac{\xi_2}{\zeta} \\ -\frac{\xi_1^2}{\zeta^2} + 4\frac{\xi_2^2}{\zeta^2} \end{pmatrix}$$

Choose $m=2$

$$= \begin{pmatrix} 2\xi_1\xi_2 - 4\xi_2\zeta \\ -\xi_1^2 + 4\xi_2^2 \end{pmatrix}$$

$$f^*(\xi_1,\xi_2,\zeta)\cdot\bar{\xi} = 2\xi_1^2\xi_2 - 4\xi_1\xi_2\zeta - \xi_1^2\xi_2 + 4\xi_2^3$$
$$= \xi_1^2\xi_2 - 4\xi_1\xi_2\zeta + 4\xi_2^3$$

$$\xi_1' = 2\xi_1\xi_2 - 4\xi_2\zeta - \xi_1^3\xi_2 + 4\xi_1^2\xi_2\zeta - 4\xi_2^3\xi_1$$
$$\xi_2' = -\xi_1^2 + 4\xi_2^2 - \xi_1^2\xi_2^2 + 4\xi_1\xi_2^2\zeta - 4\xi_2^4$$
$$\zeta' = -\xi_1^2\xi_2\zeta + 4\xi_1\xi_2\zeta^2 - 4\xi_2^3\zeta$$

$$\xi_1^2 + \xi_2^2 + \zeta^2 = 1$$

c) $\xi = 0 \implies \xi_1 \xi_2 \left(2 - \xi_1^2 - 4\xi_2^2\right) = 0$

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$$-\xi_1^2 + 4\xi_2^2 - \xi_2^2\left(\xi_1^2 + 4\xi_2^2\right) = 0$$

and $\xi_1^2 + \xi_2^2 = 1.$

$\underline{\xi_1 = 0}$, or $\xi_2 = 0$, or $\xi_1^2 + 4\xi_2^2 = 2$

$\downarrow$

$\xi_2 = \pm 1$ and $4\xi_2^2 - 4\xi_2^4 = 0 \implies (0, \pm 1, 0)$

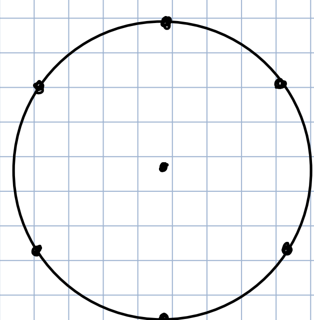
$\underline{\xi_2 = 0} \implies \xi_1 = \pm 1$ and $-1 = 0$ ↯

$\underline{\xi_1^2 + 4\xi_2^2 = 2}$ and $\xi_1^2 + \xi_2^2 = 1 \implies 3\xi_2^2 = 1$

$$\xi_2^2 = \pm \frac{1}{\sqrt{3}}$$

$\Rightarrow \xi_1^2 = \frac{2}{3}$ and $-\xi_1^2 + \frac{4}{3} - \frac{1}{3} \cdot 2 = 0$ ✓

$\xi_1 = \pm \frac{\sqrt{2}}{\sqrt{3}} \implies \left(\pm \frac{\sqrt{2}}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, 0\right)$



d)

$$f(x,y) = 2xy - 4y$$
$$g(x,y) = -x^2 + 4y^2$$

$$\dot{\bar{X}} = \bar{Z}^2\left(2\frac{\bar{X}}{\bar{Z}} \cdot \frac{1}{\bar{Z}} - 4\frac{1}{\bar{Z}}\right) - \bar{Z}^2\left(-\frac{\bar{X}^2}{\bar{Z}^2} + 4\frac{1}{\bar{Z}^2}\right)\bar{X}$$

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$$= 2\bar{X} - 4\bar{Z} + \bar{X}^3 - 4\bar{X} = -2\bar{X} - 4\bar{Z} + \bar{X}^3$$

$$\dot{\bar{Z}} = -\bar{Z}^3\left(-\frac{\bar{X}^2}{\bar{Z}^2} + 4\frac{1}{\bar{Z}^2}\right) = \bar{Z}\bar{X}^2 - 4\bar{Z}$$

Jacobian:

$$J = \begin{pmatrix} -2+3\bar{x}^2 & -4 \\ 2\bar{z}\bar{x} & -4+\bar{x}^2 \end{pmatrix}$$

Fixed pts at infinity: $(0, 1, 0)$ $\quad (\xi_1, \xi_2, \zeta)$

for $\xi_2 = 1$ $\quad \left(\pm\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$

$\bar{x} = \frac{\xi_1}{\xi_2}$, $\bar{z} = \frac{\zeta}{\xi_2}$ $\quad (\bar{x}, \bar{z})$: $\quad (0,0), (\sqrt{2}, 0), (-\sqrt{2}, 0)$

$(0,0)$: $J = \begin{pmatrix} -2 & -4 \\ 0 & -4 \end{pmatrix}$ stable node.

$(\sqrt{2}, 0)$: $J = \begin{pmatrix} 4 & -4 \\ 0 & -2 \end{pmatrix}$ saddle pt.

$(-\sqrt{2}, 0)$: $J = \begin{pmatrix} 4 & -4 \\ 0 & -2 \end{pmatrix}$ saddle pt.

e)* (extra credit)

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The symmetry in 3a) implies that if $(x(t), y(t))$ is a solution, then $(x(-t), -y(-t))$ is also a solution. Since we have uniqueness of the initial value problem this implies that solution satisfy this symmetry.

This means that in a picture of the phase plane we reflect the upper half plane in the x-axis and reverse the flow-lines in the lower half plane.

For the behavior at infinity this yields that the fixed pts at infinity $(0,-1,0)$, $\left(\pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ have the same behavior after time reversal, i.e

$(0,-1,0)$ unstable node, or $(\bar{x},\bar{z}) = (0,0)$

$\left(\pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ saddle pts, or $(\bar{x},\bar{z}) = (\mp\sqrt{2}, 0)$