Name:

Midterm 1 Department of Mathematics

Dynamical Systems 637 College of Science

Date: Tuesday March 23, 2021, 12:15 - 14:15

Instructions: 4 questions.

Please show all work and answers.

Final grade: # ptn/10.

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(1) Given the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a) [10%] Compute the eigenvalues and eigenvectors, and determine whether A is diagonalizable;

b) [15%] Determine e^{tA} and give an expression for the solution of the initial value problem starting at $t_0 = 0$;

(2) Consider the following system of differential equations:

$$\dot{x} = 2x;$$
$$\dot{y} = -y - x^3.$$

a) [10%] Find an explicit solution for the above system with initial values $x(0) = x_0$ and $y(0) = y_0$;

b) [10%] Use the answer in a) to find explicit formulas for the local stable and unstable manifolds at the equilibrium point $(0, 0)$;

c) [10%] Sketch the phase plane of flow-lines.

(3) Consider the system

$$\dot{x} = -\tfrac{3}{2}x^2 + y;$$
$$\dot{y} = -y + x.$$

a) [10%] Show that the system is a gradient system and find a potential function $V(x, y)$;

b) [10%] Compute the equilibrium points and classify them;

c) [10%] Sketch the phase plane of flow-lines (Hint: first draw some level sets for V).

(4) Consider the following system of differential equations:

$$\dot{x} = -y - x(x^2 + y^2 - 1);$$
$$\dot{y} = x - y(x^2 + y^2 - 1).$$

Denote the local flow generated by the above system by ϕ_t.

a) [10%] Show that any disc $D_r = \{(x, y) \mid x^2 + y^2 < r\}$, with $r > 1$, is forward invariant (recall that a set S is forward invariant if for every $(x, y) \in S$ there exists a time $\tau(x, y) > 0$ such that $\phi_t(x, y) \in S$ for all $t \in [0, \tau(x, y)]$);

b) [5%] Show that the circle $C = \{(x, y) \mid x^2 + y^2 = 1\}$ is invariant for the local flow ϕ_t;

Good luck!

1. a) $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. A is lower triangular form and therefore

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$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2.$

Eigenvectors: $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{pmatrix}$

$\lambda_1 = 1 \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad y = x, \quad x+y+z=0$

$z = -x-y = -2x$

$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 2$
$\lambda_3 = 2 \quad \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad x = 0, \quad x + y = 0 \Rightarrow y = 0$

$\bar{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

There is no third eigen vector. A cannot be diagonalized

b) $a_1 = e^t, \quad a_2 = \int_0^t e^{2(t-s)} e^s \, ds = e^{2t} \int_0^t e^{-s} \, ds$

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$= e^{2t}(-e^{-s}) \Big|_0^t = e^{2t} - e^t$

$a_3 = \int_0^t e^{2(t-s)} (e^{2s} - e^s) \, ds = e^{2t} \int_0^t (1 - e^{-s}) \, ds$

$= e^{2t} \int_0^t ds - a_2 = te^{2t} - e^{2t} + e^t = (t-1)e^{2t} + e^t$

$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = (A - I) I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$A_3 = (A - 2I) \cdot A_2 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

$e^{tA} = a_1(t) A_1 + a_2(t) A_2 + a_3(t) A_3$

$$e^{tA} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ e^t - e^{2t} & e^{2t} - e^t & 0 \\ e^{2t} - e^t & e^{2t} - e^t & e^{2t} - e^t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -(t-1)e^{2t} - e^t & (t-1)e^{2t} + e^t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ (2-t)e^{2t} - 2e^t & te^{2t} & e^{2t} \end{pmatrix}$$

At $t=0$, $\bar{x}(0) = \bar{x}_0 = (c_1, c_2, c_3) \Rightarrow$

$$\bar{x}(t; \bar{x}_0) = e^{tA}\bar{x}_0 = \begin{pmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ (2-t)e^{2t} - 2e^t & te^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^t \\ c_1(e^t - e^{2t}) + c_2 e^{2t} \\ c_1\left((2-t)e^{2t} - 2e^t\right) + c_2 te^{2t} + c_3 e^{2t} \end{pmatrix}$$

2. a) $\dot{x} = 2x \Longrightarrow x(t) = x_0 e^{2t}$

10 Substitute in equ. for y:

$\dot{y} = -y - x_0^3 e^{6t}$ Inhomogeneous linear diff. equ.

$$y(t) = y_0 e^{-t} + \int_0^t e^{-(t-s)}\left(-x_0^3 e^{6s}\right) ds$$

$$= y_0 e^{-t} - \int_0^t e^{-t} e^{s} x_0^3 e^{6s} ds = y_0 e^{-t} - e^{-t} x_0^3 \int_0^t e^{7s} ds$$

$$= y_0 e^{-t} - \frac{e^{-t} x_0^3}{7} e^{7s}\Big|_0^t = y_0 e^{-t} - \frac{e^{-t} x_0^3}{7}\left(e^{7t} - 1\right)$$

$$y(t) = y_0 e^{-t} + \frac{x_0^3}{7} e^{-t} - \frac{x_0^3}{7} e^{6t}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{2t} \\ y_0 e^{-t} + \frac{x_0^3}{7} e^{-t} - \frac{x_0^3}{7} e^{6t} \end{pmatrix}$$

b)

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$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{2t} \\ -\frac{x_0^3}{7} e^{6t} \end{pmatrix} + \begin{pmatrix} 0 \\ \left(y_0 + \frac{x_0^3}{7}\right) e^{-t} \end{pmatrix}$$

If $x_0 = 0 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0 e^{-t} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$

for any $(x(0), y(0)) = (0, y_0)$

loc. stable manifold $W^s_{loc}(0,0) = \{x = 0\}$

If $y_0 + \frac{x_0^3}{7} = 0 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{2t} \\ -\frac{1}{7}(x_0 e^{2t})^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

as $t \to -\infty$

for any $(x(0), y(0)) = \left(x_0, -\frac{x_0^3}{7}\right)$

loc. unstable manifold $W^u_{loc}(0,0) = \left\{y + \frac{x^3}{7} = 0\right\}$

c)

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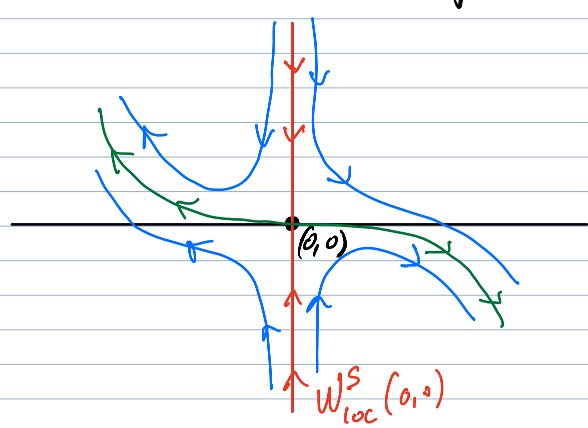

3. a) Suppose the system is a gradient system. Then there exists a potential function $V(x,y)$ such that

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$$\dot{x} = -\partial_x V$$
$$\dot{y} = -\partial_y V$$

$$-\partial_x V = -\frac{3}{2}x^2 + y \Rightarrow \partial_x V = \frac{3}{2}x^2 - y$$

$$V(x,y) = \int \left(\frac{3}{2}x^2 - y\right) dx + C(y)$$
$$= \frac{1}{2}x^3 - xy + C(y)$$
$$\partial_y V = -x + C'(y) = y - x$$
$$\Rightarrow C'(y) = y \quad \text{choose } C(y) = \frac{1}{2}y^2$$
$$V(x,y) = \frac{1}{2}x^3 - xy + \frac{1}{2}y^2$$

b)

$$-\frac{3}{2}x^2 + y = 0, \quad -y + x = 0$$
$$y = x$$

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$$\Rightarrow x = \frac{3}{2}x^2, \quad x = 0, \text{ or } x = \frac{2}{3}$$
$$y = 0 \qquad y = \frac{2}{3}$$

$$\begin{pmatrix} -\partial_{xx}^2 V & -\partial_{xy}^2 V \\ -\partial_{xy}^2 V & -\partial_{yy}^2 V \end{pmatrix} = \begin{pmatrix} -3x & 1 \\ 1 & -1 \end{pmatrix}$$

$(0,0)$:

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \det\begin{pmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = \lambda(\lambda+1) - 1 = 0$$
$$\lambda^2 + \lambda - 1 = 0$$

$$\lambda_{+/-} = \frac{-1 \pm \sqrt{1+4}}{2} \quad \begin{cases} \frac{-1+\sqrt{5}}{2} > 0 \\ \frac{-1-\sqrt{5}}{2} < 0 \end{cases} \quad \underline{\text{Saddle point}}$$

$\left(\frac{2}{3}, \frac{2}{3}\right)$: $\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, $\det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} =$

$$(\lambda+2)(\lambda+1) - 1 = \lambda^2 + 3\lambda + 1 = 0$$

$$\lambda_{+/-} = \frac{-3 \pm \sqrt{9-4}}{2} \quad \begin{cases} \frac{-3+\sqrt{5}}{2} < 0 \\ \frac{-3-\sqrt{5}}{2} < 0 \end{cases}$$

<u>stable node</u>

c) Level sets: $V(x,y) = \frac{1}{2}x^3 - xy + \frac{1}{2}y^2$

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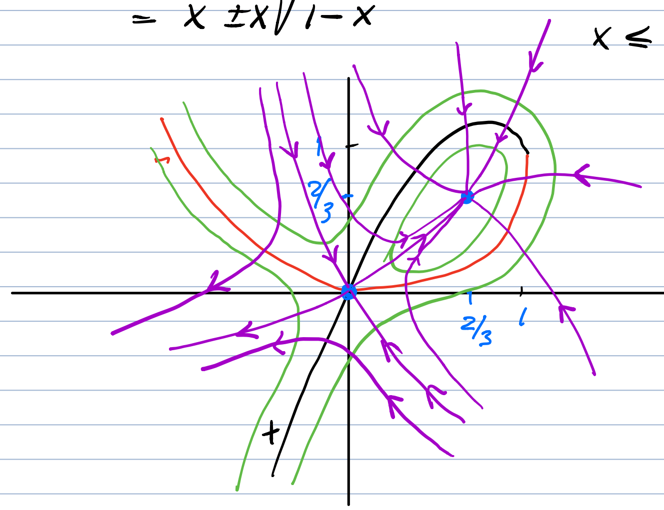
$V = 0$ $\quad y^2 - 2xy + x^3 = 0$

$$y = \frac{2x \pm \sqrt{4x^2 - 4x^3}}{2} \quad \text{two graphs}$$

$$= x \pm x\sqrt{1-x} \qquad x \leq 1$$



Flow-lines are orthogonal to the level sets of V.

4 a) $D_r = \{x^2+y^2 < r\}$, $\partial D_r = \{x^2+y^2 = r\}$

$r > 1$

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Check what ϕ_t does at ∂D_r

$$\frac{d}{dt}(x^2+y^2) = 2x\dot{x} + 2y\dot{y}$$

$$= 2x\left(-y - x(x^2+y^2-1)\right) + 2y\left(x - y(x^2+y^2-1)\right)$$

$$= -2x^2(x^2+y^2-1) - 2y^2(x^2+y^2-1)$$

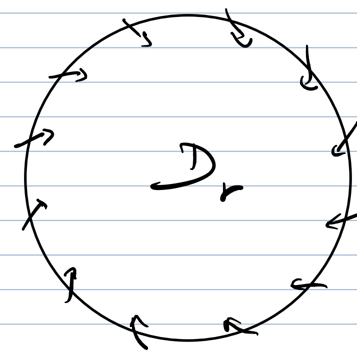
$$= -2(x^2+y^2)(x^2+y^2-1)$$

At $x^2+y^2 = r$ we have:

$$\frac{d}{dt}(x^2+y^2) = -2r\cdot(r-1) < 0$$

This proves that $(x^2+y^2)(t)$ strictly decreases at $t>0$ and therefore ϕ_t moves into the disc, i.e. $|\phi_t(\bar{x})| < r \quad \forall\, t \in (0,\tau]$.

This proves that D_r is forward invariant for all $r>1$.



b) As before

$$\frac{d}{dt}(x^2+y^2) = -2r(r-1) = 0$$

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$\quad\| \qquad$ when $r=1$

$$2x\dot{x} + 2y\dot{y} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0$$

This implies that the flow orthogonal to the gradient and thus tangent to the circle. The flow remains on the circle.