Name:

Midterm 1 Department of Mathematics

Dynamical Systems 637 College of Science

Date: Wednesday March 25, 2020, 12:15 - 14:15

Instructions: 3 questions.

Please show all work and answers.

Final grade: # ptn/10.

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(1) Given the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 9 & 5 \\ 0 & -10 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a) [10%] Compute the eigenvalues and (generalized) eigenvectors;

b) [15%] If the matrix in the above system is denoted by A, compute e^{tA} and give an expression for the solution of the initial value problem starting at $t_0 = 0$;

c) [5%] Provide a sketch of the flow lines of the above system.

(2) Consider the following system of differential equations:

$$\dot{x} = -x;$$
$$\dot{y} = y - x^3 + x^2.$$

a) [5%] Compute all equilibrium points;

b) [10%] Determine eigenvalues and eigenvectors of the point $(0, 0)$;

In order to compute the local stable manifold (curve) of the equilibrium point $(0, 0)$ we use the iteration scheme:

$$\mathbf{u}^{n+1}(t, \mathbf{a}) = U(t)\mathbf{a} + \int_0^t U(t-s)\mathbf{G}(\mathbf{u}^n(s, \mathbf{a}))ds - \int_t^\infty V(t-s)\mathbf{G}(\mathbf{u}^n(s, \mathbf{a}))ds$$

$$\mathbf{u}^0(t, \mathbf{a}) = (0, 0),$$

where $U(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}$ and $V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^t \end{pmatrix}$ and $\mathbf{G}(x, y) = \begin{pmatrix} 0 \\ -x^3 + x^2 \end{pmatrix}$.

c) [15%] Carry out the iteration scheme and show that the sequence stabilizes (hint: use $\mathbf{a} = (a_1, 0)$ and determine $\mathbf{u}(t, \mathbf{a})$);

d) [10%] Derive an equation for the local stable manifold;

e) [10%][1] Explain why the equation for the local stable manifold yields the global stable manifold of $(0, 0)$.

(3) Consider the system

$$\dot{x} = y;$$
$$\dot{y} = x^3 - x^2 - 2x.$$

a) [10%] Show that the system is Hamiltonian and find a Hamiltonian;

b) [10%] Compute the equilibrium points and determine their nature;

c) [10%] Sketch the phase plane of flow lines.

Good luck!

[1] Extra credit problem.

Answers Midterm 1, March 25, 2020.

1. a)

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 0 & 0 \\ 0 & 9-\lambda & 5 \\ 0 & -10 & -5-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = -(1+\lambda)\cdot\left[(9-\lambda)(-5-\lambda)+50\right]$$
$$= -(1+\lambda)\left[(\lambda-9)(\lambda+5)+50\right] = -(1+\lambda)\left[\lambda^2-4\lambda+5\right]$$
$$= -(1+\lambda)\left((\lambda-2)^2+1\right) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 2+i, \quad \lambda_3 = 2-i$$

$\lambda_1 = -1$:

$$\left.\begin{aligned} 10y + 5z &= 0 \\ -10y - 4z &= 0 \end{aligned}\right\} \Rightarrow y = z = 0$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_2 = 2+i$:

$$(-3-i)x = 0 \quad \Rightarrow x = 0$$
$$(7-i)y + 5z = 0$$
$$-10y - (7+i)z = 0$$

$$\begin{aligned} 10y &= -(7+i)z \\ 10z &= 10z \end{aligned} \quad \Rightarrow v_2 = \begin{pmatrix} 0 \\ -(7+i) \\ 10 \end{pmatrix}$$

$\lambda_3 = 2-i$: $\quad v_3 = \overline{v_2}$.

b) $A = PDP^{-1}$, where

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$P = \left(v_1 \quad \operatorname{Im} v_2 \quad \operatorname{Re} v_2 \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -7 \\ 0 & 0 & 10 \end{pmatrix}$$

For P^{-1} it suffices to compute $\begin{pmatrix} -1 & -7 \\ 0 & 10 \end{pmatrix}^{-1}$

$$= -\frac{1}{10} \begin{pmatrix} 10 & 7 \\ 0 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -7/10 \\ 0 & 0 & 1/10 \end{pmatrix}$$

$$e^{tA} = Pe^{tD}P^{-1} = P \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}\cos t \end{pmatrix} P^{-1}$$

We have to compute

$$\frac{e^{2t}}{10} \begin{pmatrix} -1 & -7 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -10 & -7 \\ 0 & 1 \end{pmatrix}$$

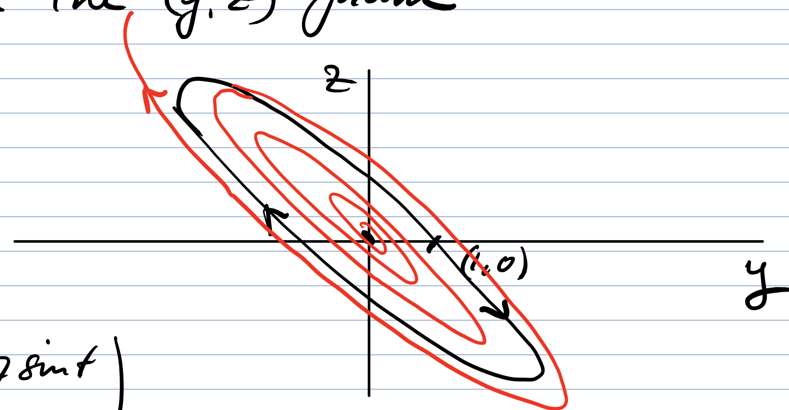
$$= \frac{e^{2t}}{10} \cdot 10 \begin{pmatrix} \cos t + 7\sin t & 5\sin t \\ -10\sin t & \cos t - 7\sin t \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} \cos t + 7\sin t & 5\sin t \\ -10\sin t & \cos t - 7\sin t \end{pmatrix}$$

$$\Rightarrow \quad e^{tA} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t}\cos t + 7e^{2t}\sin t & 5e^{2t}\sin t \\ 0 & -10e^{2t}\sin t & e^{2t}\cos t - 7e^{2t}\sin t \end{pmatrix}$$

$$\bar{X}(t) = e^{tA}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} e^{-t} x_0 \\ y_0 e^{2t}\cos t + (7y_0 + 5z_0)e^{2t}\sin t \\ (-10y_0 - 7z_0)e^{2t}\sin t + z_0 e^{2t}\cos t \end{pmatrix}$$

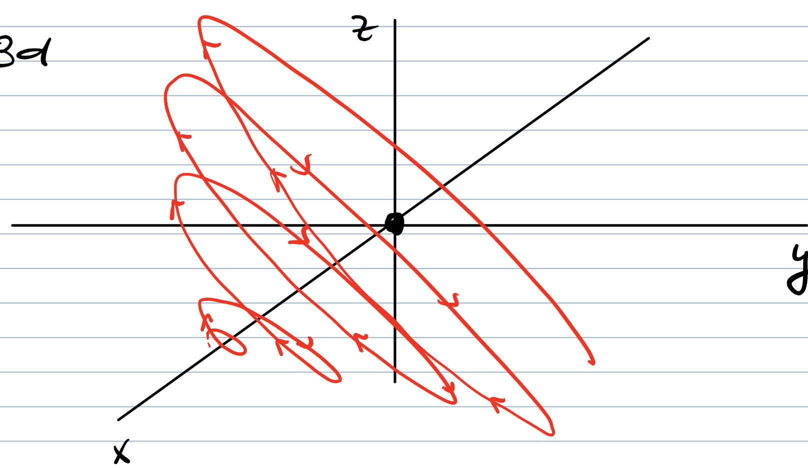
c) In the (y,z)-plane



$\begin{pmatrix} \cos t + 7\sin t \\ -10\sin t \end{pmatrix} \nearrow$ ellipse

the ellipse is the base.

In 3d



2. a) $-x = 0 \implies x = 0$

$y - x^3 + x^2 = 0$

$\implies y = 0$ $\quad$ $(0,0)$ is the only equilibrium pt.

b) $\bar{f}(x,y) = \begin{pmatrix} -x \\ y - x^3 + x^2 \end{pmatrix}$

$$D\bar{f}(x,y) = \begin{pmatrix} -1 & 0 \\ -3x^2+2x & 1 \end{pmatrix}, \quad D\bar{f}(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues are on the diagonal $\lambda_1 = -1, \lambda_2 = 1$

Eigenvectors unit basis since the matrix is diagonal

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The point $(0,0)$ is a saddle pt. and the coordinate axes are the linear stable and unstable spaces.

c) $\bar{a} = (a_1, 0)$, $u^0(t,\bar{a}) = (0,0)$

$$G(x,y) = \begin{pmatrix} 0 \\ -x^3+x^2 \end{pmatrix} \Rightarrow G(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u^1(t,\bar{a}) = U(t)\,\bar{a} = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix}$$

$$u^2(t,\bar{a}) = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix} \begin{pmatrix} 0 \\ -e^{-3s} a_1^3 + e^{-2s} a_1^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 \\ e^{t}\left(-a_1^3 e^{-4s} + a_1^2 e^{-3s}\right) ds \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} a_1 \\ e^{t}\left[a_1^3 \frac{-1}{4} e^{-4s} + a_1^2 \frac{1}{3} e^{-3s} \right]_t^\infty \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} a_1 \\ \frac{a_1^3}{4} e^{-3t} - \frac{a_1^2}{3} e^{-2t} \end{pmatrix}$$

$$u^3(t,\bar a) = \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix} \begin{pmatrix} 0 \\ -e^{-3s}a_1^3 + e^{-2s}a_1^2 \end{pmatrix} ds$$

$$= u^2(t,\bar a)$$

$\Longrightarrow$ iteration stabilizes at u^2 and

$u^n(t,\bar a) = u^2(t,\bar a), \ \forall n \geq 2$

$$\Longrightarrow \quad u(t,\bar a) = u^2(t,\bar a)$$

$$= \begin{pmatrix} e^{-t}a_1 \\ \frac{a_1^3}{4}e^{-3t} - \frac{a_1^2}{3}e^{-2t} \end{pmatrix}$$

d) $u_1(0,\bar a) = a_1$ initial x-coordinate

$u_2(0,\bar a) = \frac{a_1^3}{4} - \frac{a_1^2}{3}$ y-coordinate as function of the x-coord.

$\Longrightarrow$ Stable manifold (curve) is given as local graph of the x-axis

$$\psi(x) = \frac{x^3}{4} - \frac{x^2}{3}$$

e) (Extra) $S = \{(x, \psi(x)), \ x \in \mathbb{R}\}$

is an invariant set for the flow:

at a pt $(x, \psi(x))$, the tangent vector is given by

$$\begin{pmatrix} 1 \\ \psi'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4}x^2 - \frac{2}{3}x \end{pmatrix}$$

Claim: this matches with the vector field.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-x\\ y - x^3 + x^2\end{pmatrix} = \begin{pmatrix}-x\\ \frac{1}{4}x^3 - \frac{1}{3}x^2 - x^3 + x^2\end{pmatrix}$$

$y = \psi(x)$

$$= \begin{pmatrix}-x\\ -\frac{3}{4}x^3 + \frac{2}{3}x^2\end{pmatrix} = -x\begin{pmatrix}1\\ \frac{3}{4}x^2 - \frac{2}{3}x\end{pmatrix}$$

$\Longrightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ is tangent to S.

Initial pts on S 'flow' to $(0,0)$ on S

3. a)

$$\dot{x} = y = H_y$$

$$\dot{y} = x^3 - x^2 - 2x = -H_x$$

$$H = \int H_y + C(x) = \frac{1}{2}y^2 + C(x)$$

$$H_x = C'(x) = -x^3 + x^2 + 2x$$

$$\Longrightarrow \quad C(x) = \int -x^3 + x^2 + 2x \quad + C$$

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2$$

take $C = 0$.

This prove that the system is Hamiltonian and H is a Hamiltonian.

b) $y = 0$, $\quad x^3 - x^2 - 2x = 0$

$$\Updownarrow$$

$$x(x^2 - x - 2) = x(x-2)(x+1) = 0$$

Pts: $(-1,0)$, $(0,0)$, $(2,0)$

$$\bar{f}(x,y) = \begin{pmatrix} y \\ x^3 - x^2 - 2x \end{pmatrix}, \quad D\bar{f}(x,y) = \begin{pmatrix} 0 & 1 \\ 3x^2 - 2x - 2 & 0 \end{pmatrix}$$

$(-1,0)$: $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \Rightarrow \lambda^2 - 3 = 0$

$\lambda_1 = -\sqrt{3}, \quad \lambda_2 = \sqrt{3}$

saddle pt

$(0,0)$: $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow \lambda^2 + 2 = 0$

$\lambda_1 = i\sqrt{2}, \quad \lambda_2 = -i\sqrt{2}$

center

$(2,0)$: $\begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix} \Rightarrow \lambda^2 - 6 = 0$

$\lambda_1 = -\sqrt{6}, \quad \lambda_2 = \sqrt{6}$

saddle pt.

c) Hamiltonian is preserved along flow lines

$\Longrightarrow$

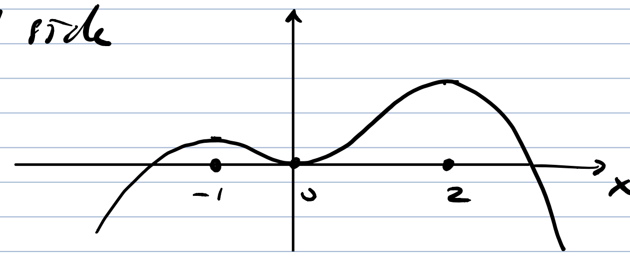
$$\frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 = E$$

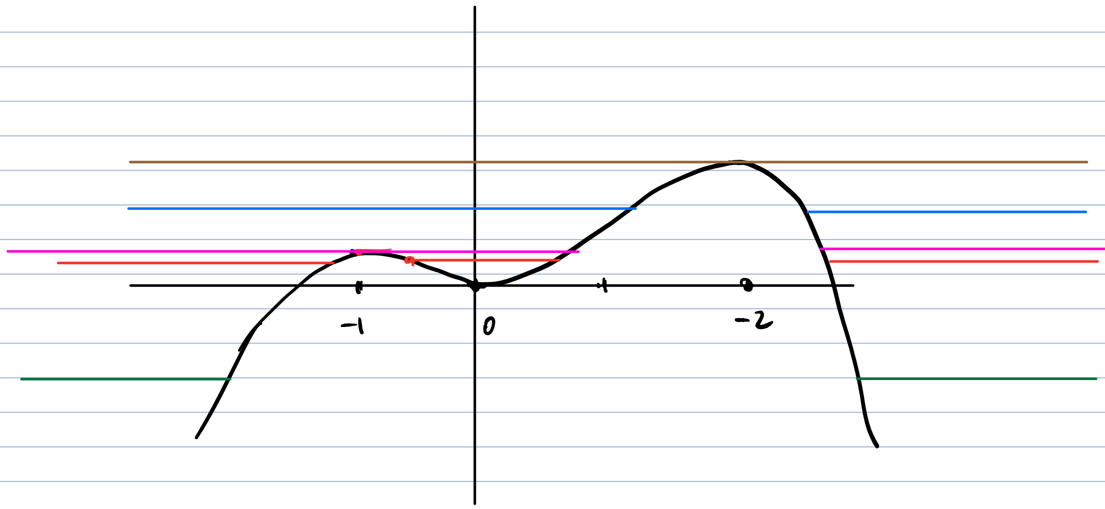
$$\Longrightarrow \quad -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 \le E$$

for given energy value E

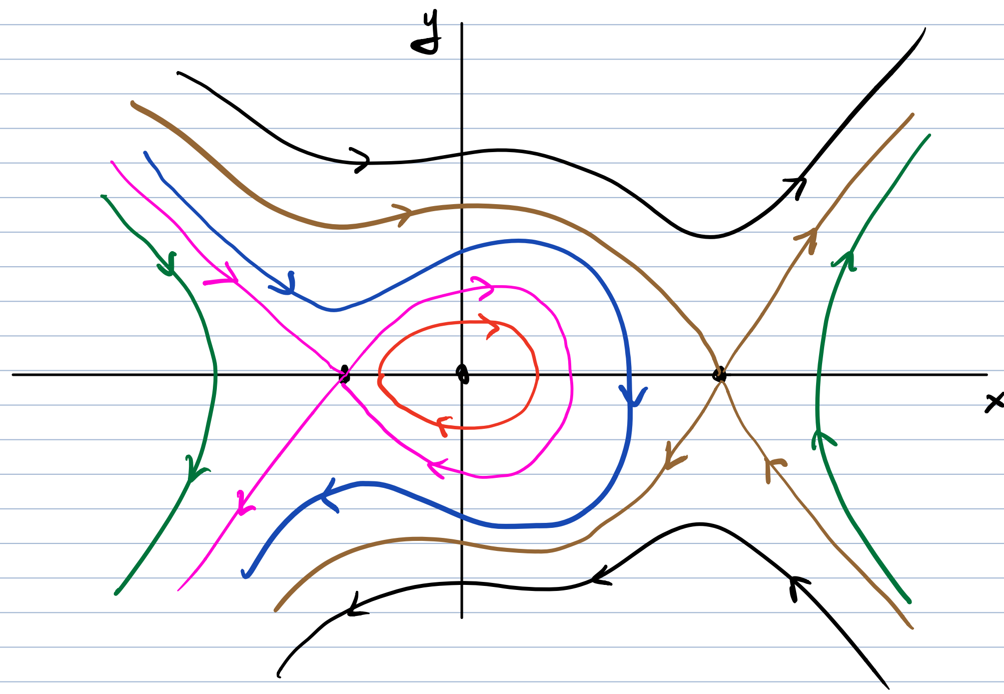
Graph of the left hand side

potential





-1
0
1
2



y
x