Final Exam | **XB_0008: Discrete Mathematics**
VU Amsterdam | **22 December 2022 (15:30-17:30)**

Please justify your answers! Even a correct answer without full explanation scores badly.

The use of books, lecture notes, calculators, etc. is not allowed.

Question 1. Consider the symmetric group S_8.

(a) Write $\sigma \in S_8$ as a product of disjoint cycles where $\sigma = (276534)(14)(18574)(28)(1537246)$.

(b) Write $\sigma^{-3} = (\sigma^{-1})^3$ as a product of disjoint cycles where σ^{-1} is the inverse of σ.

`Note: No explanation is necessary for final answers in parts (a)--(b) of this question.`

Question 2. Show that for all $n \in \mathbb{Z}_{\geq 0}$

$$\sum_{a+b+c+d=n} \binom{n}{a,b,c,d} (-1)^{a+b} x^c = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k}.$$

Question 3. Consider the symmetric group S_6 with the identity element denoted by e. Determine the number of permutations $\pi \in S_6$ such that $\pi^6 = e$ and π is either a 6-cycle or can be written as a disjoint product of a 3-cycle and *at least* one other (nontrivial) cycle.

Question 4. Show that the number of permutations on n objects that contain exactly one cycle of length 1 is given by

$$n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}.$$

`Note: You are expected to derive this formula and verify each step in your argument. Note that you are not allowed to use the derangement formula directly without giving an explicit proof.`

Question 5. Let $a_0 = 5$ and consider the recurrence relation $a_k = a_{k-1} + 2k$ for $k \in \mathbb{Z}_{>0}$.

(a) Show that the generating function $G(x)$ for the sequence $\{a_k\}$ satisfies

$$G(x) = \frac{5}{1-x} + \frac{2x}{(1-x)^3}.$$

(b) Determine a (simple closed) formula for a_k ($k \in \mathbb{Z}_{\geq 0}$).
`Note: You may use the equality given in (a) above for answering (b).`

Question 6. (a) Write down the elements of the cyclic group C_8 as a product of disjoint cycles.

(b) How many ways are there to colour the vertices of an octagon (i.e. a regular 8-sided polygon) if k different colours are available, where k is a positive integer, discounting (planar) *rotational* symmetries?

Maximum score per subitem

1a: 5 1b: 5	2: 10	3: 12	4: 18	5a: 10 5b: 10	6a: 5 6b: 15

Maximum Total = 90

Mark = 1 + (Total/10)