Discrete Mathematics - Final exam 2022: Solutions

(1) (a) We have $\sigma = (276534)(14)(18574)(28)(1537246) = (16873)(45)$.
(b) We have $\sigma^{-3} = (\sigma^{-1})^3 = ((16873)^{-1}(45)^{-1})^3 = (13786)^3(45)^3 = (18367)(45)$.

(2)

$$\begin{aligned}\sum_{a+b+c+d=n}\binom{n}{a,b,c,d}(-1)^{a+b}x^c &= \sum_{a+b+c+d=n}\binom{n}{a,b,c,d}(-1)^a(-1)^b x^c 1^d \\ &\overset{\text{multinomial theorem}}{=} (-1-1+x+1)^n = (x-1)^n \\ &\overset{\text{binomial theorem}}{=} \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n-k}.\end{aligned}$$

(3) The number of 6-cycles in S_6 is $\frac{6!}{6} = 5! = 120$. If π is a product of a 3-cycle and another disjoint cycle, then π is either a product of two disjoint 3-cycles or a disjoint product of a 3-cycle and a 2-cycle. In the former, there are $\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot3\cdot2} = 40$ and in the latter $\frac{6\cdot5\cdot4\cdot3\cdot2}{3\cdot2} = 120$ such permutations. In total there are $120+40+120 = 280$ such π.

(4) Having a cycle of length 1 means a fixed position for a permutation. Let π be a permutation such that $\pi(k) = k$ and $\pi(i) \neq i$ for $i \neq k$. Then π is a derangement of $n-1$ elements. The total number of such a π, i.e. the derangement of a set of $n-1$ elements, is given by $(n-1)¡$. If we run k through $1, \dots, n$, we obtain the total number of permutations that contain exactly one cycle is $n(n-1)¡$.

Let us compute $(n-1)¡$. We have $N = (n-1)!$. Let a_i be the property "$\pi(i) = i$", a_ia_j be the property "$\pi(i) = i$ and $\pi(j) = j$", and so on. Then $N(a_i) = (n-2)!$, $N(a_ia_j) = (n-3)!$, and so on. By the principle of inclusion and exclusion,

$$\begin{aligned}(n-1)¡ &\overset{3}{=} N - \sum_{i=1}^{n-1}N(a_i) + \sum_{1\le i<j\le n-1}N(a_ia_j) - \dots + (-1)^r\sum_{1\le i_1<\dots<i_r\le n-1}N(a_{i_1}\dots a_{i_r})\dots + (-1)^{n-1}N(a_1\dots a_{n-1}) \\ &\overset{2}{=} (n-1)! - \sum_{i=1}^{n-1}(n-1)! + \sum_{1\le i<j\le n-1}(n-3)! - \dots + (-1)^r\sum_{1\le i_1<\dots<i_r\le n-1}(n-r-1)!\dots + (-1)^{n-1} \\ &\overset{2}{=} (n-1)! - \binom{n-1}{1}(n-2)! + \binom{n-1}{2}(n-3)! - \dots + (-1)^r\binom{n-1}{r}(n-r-1)! + \dots + (-1)^{n-1}\binom{n-1}{n-1} \\ &\overset{1}{=} \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k-1)! = \sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!}{k!(n-k-1)!}(n-k-1)!\end{aligned}$$

and thus $(n-1)¡ = (n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}$. Hence the total number of permutations containing exactly one cycle of length 1 is given by $n(n-1)¡ = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$.

(5) (a) For the recurrence relation $a_k = a_{k-1} + 2k$ for $k \in \mathbb{Z}^+$ and with the initial condition $a_0 = 5$, the generating function is given by

$$\begin{aligned} G(x) &= \sum_{k\geq 0} a_k x^k = a_0 + \sum_{k\geq 1} a_k x^k \\ &= 5 + \sum_{k\geq 1} a_{k-1}x^k + \sum_{k\geq 1} 2kx^k \\ &= 5 + x\sum_{k\geq 0} a_k x^k + 2x\sum_{k\geq 0} kx^{k-1} \\ &= 5 + xG(x) + \frac{2x}{(1-x)^2} \\ (1-x)G(x) &= 5 + \frac{2x}{(1-x)^2} \\ G(x) &= \frac{5}{1-x} + \frac{2x}{(1-x)^3} \end{aligned}$$

(b) The geometric series is given by $\sum_{k\geq 0} x^k = \frac{1}{1-x}$ and by differentiating we obtain $\sum_{k\geq 1} kx^{k-1} = \frac{1}{(1-x)^2}$ and $\sum_{k\geq 2} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$. Therefore,

$$\frac{2x}{(1-x)^3} = \sum_{k\geq 2} k(k-1)x^{k-1} = \sum_{k\geq 1}(k+1)kx^k = \sum_{k\geq 0} k(k+1)x^k$$

. Since $\frac{5}{1-x} = 5\sum_{k\geq 0} x^k$, we have

$$G(x) = 5\sum_{k\geq 0} x^k + \sum_{k\geq 0} k(k+1)x^k = \sum_{k\geq 0}[5 + k(k+1)]x^k$$

and thus $a_k = 5 + k(k+1)$ for $k \in \mathbb{Z}^{\geq 0}$

(6) (a) We have $C_8 = \langle\sigma\rangle = \{\sigma^m : m \in \mathbb{Z}^{\geq 0}\}$, where $\sigma = (12345678)$. Thus, we obtain

$$C_8 = \{e, (12345678), (1357)(2468), (14725836), (15)(37)(26)(48), (16385274), (1753)(2864), (18765432)\}$$

(b) We have $|G| = |C_8| = 8$ and by Burnside's lemma, $N = \#\ \textit{orbits} = \frac{1}{|G|}\sum_{\pi\in G}|C_\pi|$. Since $|C_1| = k^8$, $|C_\sigma| = |C_{\sigma^3}| = |C_{\sigma^5}| = |C_{\sigma^7}| = k$, $|C_{\sigma^2}| = |C_{\sigma^6}| = k^2$ and $|C_{\sigma^4}| = k^4$. Thus $N = \frac{k^8+4k+2k^2+k^4}{8}$.