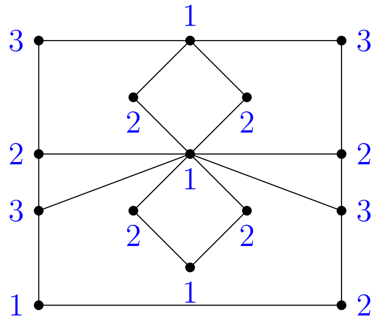
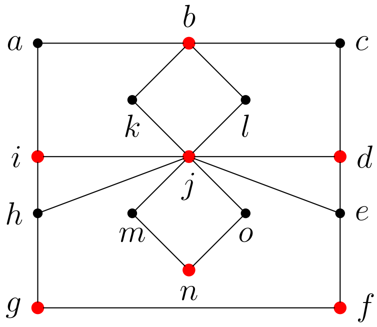
Discrete Mathematics - Midterm exam 2022: Solutions

(1) (a) A graph contains an Eulerian trail if and only if it contains at most two vertices of odd degree. Since G has more than two vertices of odd degree (namely, i, h, d, e), it does not contain an Eulerian trail.

(b) A graph is bipartite if and only if it doesn't contain any odd cycles. Since G contains odd cycles (e.g. h, i, j), it is not bipartite.

(c) Notice that since G contains a 3-cycle, its chromatic number is at least three. Since there exists a 3-colouring of G (e.g. see figure below left), $\chi(G) = 3$.

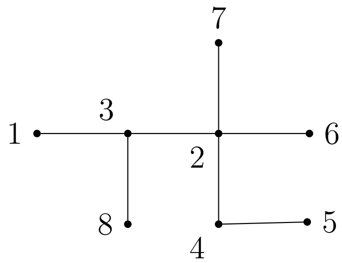




(d) E.g., the vertices $\{b, d, f, g, i, j, n\}$ in the figure above on the right is an edge cover since all edges in G are incident with at least one of the vertices in this collection and it has size 7.

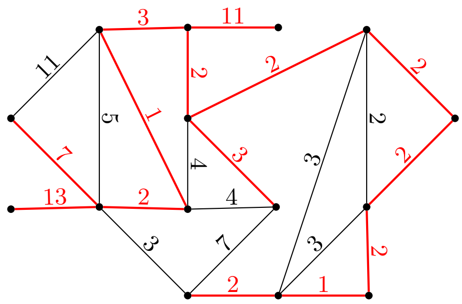
(e) Notice that $j \to A, m \to B, n \to C, o \to D, k \to E, b \to F, l \to G, a \to I, i \to H, c \to J, d \to K, h \to M, e \to L, g \to N, f \to O$ is a bijection of the vertices of G and H that preserves adjacency, thus G and H are isomorphic graphs.

(2) (a) The labelled tree corresponding to the Prüfer sequence $3, 4, 2, 2, 2, 3$ is given by



and justification.

(b)



a correct spanning tree and justification. The weight of a minimal spanning tree is $2 \times 1 + 7 \times 2 + 2 \times 3 + 7 + 11 + 13 = 53$.

(3) (a) Let $c = \sum_R b(R)$, where $b(R)$ is the bound degree of a region R. Since every edge bounds at most two regions, we have $c \leq 2q$. Moreover, $c \geq 8(r-2)+10+12 = 8r+6$ since a planar representation of G has two regions one is bounded by 10 edges and

the other by 12, and all other regions have bound degree at least 8 since G is bipartite (so no odd cycles).. Then $2q \geq 8r + 6$ and $q \geq 4r + 3$ as required.

(b) By Euler's formula, $n - q + r = 2$ where n is the order, q is the size of G and r is the number of regions in any planar representation of G. Then $r = 2 - n + q$ and substituting in part (a), we obtain $q \geq 4(2 - n + q) + 3 = 4q - 4n + 11$ and thus $4n \geq 3q + 11$.

(4) Suppose G is of order n. If G is connected, then it has one connected component. Any spanning tree of G has $n - 1$ edges so G has size at least $n - 1$. Therefore, $q + c \geq (n-1)+1 = n$ and the inequality holds. Now assume G is not connected. Let $X_1, \ldots, X_c$ be connected components of G of order $n_1, \ldots, n_c$ and has size $q_1, \ldots, q_c$, respectively. Then $n = n_1 + \cdots + n_c$ and the size of G is $q = q_1 + \cdots + q_c$. By the argument above, for each i, where $1 \leq i \leq c$, we have $q_i + 1 \geq n_i$. Then $n = n_1 + \ldots n_c \leq (q_1 + 1) + \cdots + (q_c + 1) = (q_1 + \cdots + q_c) + c = q + c$ where c is the number of connected components, as required.

(5) Suppose G is 4-regular. Assume first G is connected. Then every vertex of G is of order 4, i.e. is even. Therefore, G is Eulerian and admits an Eulerian circuit C, say. Suppose we transverse the edges in C in the order $a_1, a_2, \ldots, a_t$. We colour successive edges in C by alternate colours. Then every time there are successive edges a_i, a_{i+1} in C that are incident with a vertex x, we use two different colours. Since the degree of x is 4 for any $x \in V(G)$, there are 4 incident vertices with x and we use each colour once for successive pairs in C so we colour precisely two edges with the same colour and the other two with the other colour, as required. If G is not connected, then each connected component is still 4-regular and we may argue as above for each connected component of G.