Final Exam | **XB_0008: Discrete Mathematics**
VU Amsterdam | **23 December 2021 (15:30-17:30)**

Please justify your answers! Even a correct answer without full explanation scores badly.

The use of books, lecture notes, calculators, etc. is not allowed.

Question 1. Consider the symmetric group S_8.

(a) Write $\sigma \in S_8$ as a product of disjoint cycles where $\sigma = (1827634)(37)(27653)(148526)(37)$.
(b) Write σ^3 as a product of disjoint cycles.
(c) Find the inverse of σ as a product of disjoint cycles.

`Note: No explanation is necessary for final answers in this question.`

Question 2. Show that for all $n \in \mathbb{Z}_{\geq 0}$

$$\sum_{a+b+c+d=n} \binom{n}{a,b,c,d}(-1)^a x^c = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Question 3. Consider the symmetric group S_7 with the identity element denoted by e. Determine the number of permutations $\pi \in S_7$ that satisfy $\pi^6 = e$ and which can be written as a disjoint product of a 3-cycle and *at least* one other (nontrivial) cycle.

Question 4. Let π be a permutation of $\{1, 2, ..., n\}$ such that π only fixes a single position, that is there exists some k with $1 \leq k \leq n$ such that $\pi(k) = k$ and $\pi(i) \neq i$ for $i \neq k$. Verify that the number of such π is given by

$$n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}.$$

`Note: You are not allowed to use the formula for the number of derangements in this question. Please verify every step in your argument.`

Question 5. Consider the sequence $\{a_k\}_{k \in \mathbb{Z}_{\geq 0}}$ recursively defined by

$$a_0 = 2, \quad a_1 = 8, \quad a_k = 8a_{k-1} - 15a_{k-2}, \quad k \in \mathbb{Z}_{\geq 2}.$$

(a) Show that the generating function $G(x)$ for the sequence $\{a_k\}$ ($k \in \mathbb{Z}_{\geq 0}$) satisfies

$$(15x^2 - 8x + 1)G(x) = 2 - 8x.$$

(b) Determine a (simple closed) formula for a_k for $k \in \mathbb{Z}_{\geq 0}$.

`Note: You may use the equality given in part (a) for answering part (b).`

Question 6. How many ways are there to colour the vertices of a regular pentagon (i.e. a regular 5-sided polygon) if k different colours are available, where k is a positive integer, discounting (planar) *rotational* and *reflectional* symmetries?

Maximum score per subitem

1a: 5 1b: 4 1c: 3	2: 10	3: 14	4: 16	5a: 10 5b: 10	6: 18

Maximum Total = 90

Mark = 1 + (Total/10)