2021 Discrete Mathematics – Final Exam Solutions

① ⓐ $\sigma = (1\,8\,2\,7\,6\,3\,4)(3\,7)(2\,7\,6\,5\,3)(1\,4\,8\,5\,2\,6)(3\,7)$

$= (2\,5\,4)(6\,8)$

ⓑ $\sigma^3 = (2\,5\,4)^3(6\,8)^3 = (6\,8)$

ⓒ $\sigma^{-1} = (2\,4\,5)(6\,8)$

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② Let $n \in \mathbb{Z}_{\geq 0}$,

$$\sum_{a+b+c+d=n} \binom{n}{a,b,c,d} (-1)^a x^c = \sum_{a+b+c+d=n} \binom{n}{a,b,c,d} (-1)^a 1^b x^c 1^d$$

$$= (-1+1+x+1)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

by multinomial theorem

by binomial theorem

③ $\pi \in S_7$ product of at least two cycles and

$$\pi^6 = e \xrightarrow{\text{I}} \pi \text{ is a product of two 3-cycles}$$

$$\pi^6 = e \xrightarrow{\text{II}} \pi \text{ is a product of a 3-cycle and a transposition}$$

$$\pi^6 = e \xrightarrow{\text{III}} \pi \text{ is a product of two transpositions and a 3-cycle}$$

$$\#(\text{case I}) = \frac{7 \cdot \cancel{6} \cdot 5 \cdot 4 \cdot \cancel{3} \cdot 2}{\cancel{3} \cdot \cancel{3} \cdot \cancel{2}} = 280$$

$$\#(\text{case II}) = \frac{7 \cdot \cancel{6} \cdot 5 \cdot 4 \cdot 3}{\cancel{3} \cdot \cancel{2}} = 420$$

$$\#(\text{case III}) = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot 2 \cdot 1}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{2}} = 210$$

$$\text{Total} = 280 + 420 + 210 = 910$$

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④ Let π be a permutation that fixes only the k^{th} position for some $1 \le k \le n$; i.e. $\pi(k) = k$ and $\pi(i) \neq i$ for $i \neq k$. Then π is a derangement of $n-1$ elements. The total number of such a π is the derangement of a set of $n-1$ elements given by $(n-1)¡$. Since k runs over $\{1, 2, \dots, n\}$, altogether there are $\boxed{n(n-1)¡}$

permutations fixing only a single position.

Let us compute $(n-1)_i$. We have that $|N| = |S_{n-1}| = (n-1)!$. Let

a_i = "element i is fixed in a permutation" (i.e. $\pi(i) = i$)

Then $N(a_i) = (n-2)!$, $N(a_i a_j) = (n-3)!$, ...
$N(a_1 a_2 \dots a_{n-1}) = 1$

By the principle of Inclusion-Exclusion,

$$(n-1)_i = N - \sum_{i=1}^{n-1} N(a_i) + \sum_{1 \leq i < j \leq n-1} N(a_i a_j) - \dots$$

$$+ (-1)^r \sum_{1 < i_1 < \dots < i_r \leq n-1} N(a_{i_1} \dots a_{i_r}) + \dots + (-1)^{n-1} N(a_1 \dots a_{n-1})$$

$$= (n-1)! - \sum_{i=1}^{n-1} (n-2)! + \sum_{1 \leq i < j \leq n-1} (n-3)! + \dots + (-1)^r \sum (n-r-1)! \dots + (-1)^{n-1}$$

$$= (n-1)! - \binom{n-1}{1}(n-2)! + \binom{n-1}{2}(n-3)! + \dots +$$

$$+ (-1)^r \binom{n-1}{r}(n-r-1)! + \dots + (-1)^{n-1}\binom{n-1}{n-1}$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k-1)!$$

$$= \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!\,\cancel{(n-1-k)!}} \cancel{(n-k-1)!}$$

$$(n-1)_i = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}$$

Therefore

$$n(n-1)_i = n(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}.$$

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5 a $a_0 = 2$, $a_1 = 8$, $a_k = 8a_{k-1} - 15a_{k-2}$, $k \in \mathbb{Z}_{\geq 2}$

$$G(x) = \sum_{k \geq 0} a_k x^k$$

$$= a_0 + a_1 x + \sum_{k \geq 2} a_k x^k$$

$$= 2 + 8x + \sum_{k \geq 2} (8a_{k-1} - 15a_{k-2}) x^k$$

$$= 2 + 8x + 8 \sum_{k \geq 2} a_{k-1} x^k - 15 \sum_{k \geq 2} a_{k-2} x^k$$

$$= 2 + 8x + 8x \sum_{k \geq 1} a_k x^k - 15x^2 \sum_{k \geq 0} a_k x^k$$

$$G(x) = 2 + 8x + 8x\left(G(x) - 2 \right) - 15x^2 G(x)$$

$$\left(15x^2 - 8x + 1 \right) G(x) = 2 - 8x$$

b $$G(x) = \frac{2-8x}{15x^2-8x+1} = \frac{2-8x}{(5x-1)(3x-1)} = \frac{A}{5x-1} + \frac{B}{3x-1}$$

$$2 - 8x = A(3x-1) + B(5x-1)$$

$$x = 1/5 \implies 2 - \frac{8}{5} = A\left(\frac{3}{5} - 1\right) \implies A = -1$$

$$x = 1/3 \implies 2 - \frac{8}{3} = B\left(\frac{5}{3} - 1\right) \implies B = -1$$

$$G(x) = \frac{1}{1-5x} + \frac{1}{1-3x}$$

$$= \sum_{k \geq 0} (5x)^k + \sum_{k \geq 0} (3x)^k = \sum_{k \geq 0} (5^k + 3^k) x^k$$

$$\implies \boxed{a_k = 5^k + 3^k} \quad \text{for } k \geq 0$$

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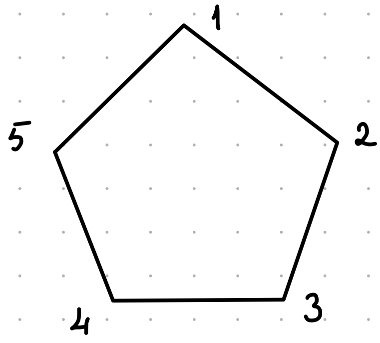

$$G = D_5 = \{ e, (12345), (13524), (14253), (15432), (25)(34), (13)(45), (15)(24), (12)(35), (14)(23) \}$$

$$|G| = |D_5| = 10$$

$$N = \#\text{orbits} = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi| \quad \text{by Burnside's lemma}$$

$$|C_e| = k^5$$

$$|C_{(12345)}| = |C_{(13524)}| = |C_{(14253)}| = |C_{(15432)}| = k$$

$$|C_{(25)(34)}| = |C_{(13)(45)}| = |C_{(15)(24)}| = |C_{(12)(35)}| = |C_{(14)(23)}| = k^3$$

$$N = \frac{k^5 + 5k^3 + 4k}{10}$$