Discrete Mathematics

2020 Midterm – Solutions

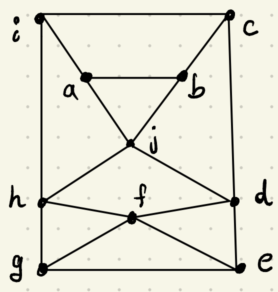
1. a) Notice that G has more than 2 vertices of odd degree, e.g. $a, b, c, e, \ldots$ Since a connected graph has an Eulerian trail if and only if it contains at most 2 vertices of odd degree, G does not contain an Eulerian trail.

b) G and H are isomorphic graphs if there exists a bijection $f: V(G) \to V(H)$ such that for each $x, y \in V(G)$, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$.

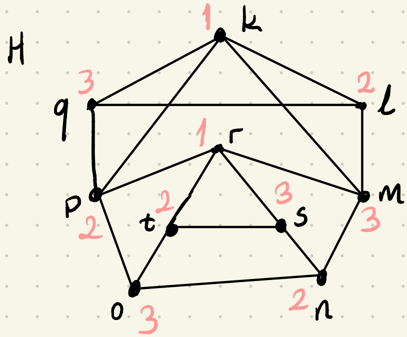
Consider $f = \{(f,k), (j,r), (g,q), (e,l), (h,p), (d,m), (i,o), (c,n), (a,t), (b,s)\}$

Then G and H are isomorphic.

c) Yes.



d)

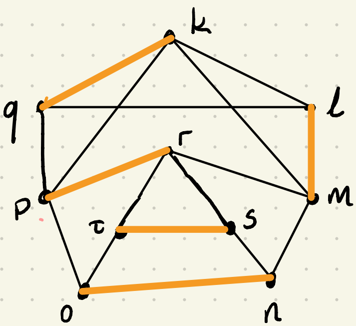


So $G\,(\cong H)$ is 3-colourable.

Is $G\,(\cong H)$ 2-colourable?

No, because otherwise, we could colour the vertices k, p, q using 2 colours. But these vertices form a 3-cycle. Hence $\chi(G) = 2$.

e)

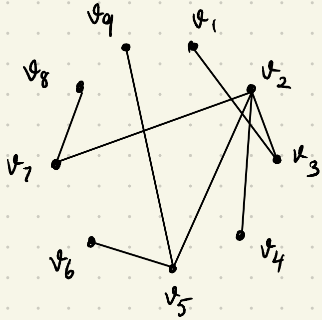


2 Suppose $\deg(v)$ is odd. Let G' be the connected component of G containing v. The sum of the degrees of the vertices in G' is even, so there must be another vertex w in G' which has odd degree. Since v and w are in the same connected component, there must be a path from v to w.

(3) Since $n \geq 3$ and $\delta(G) \geq n/2$, G is Hamiltonian so it has a Hamiltonian circuit, say C. Label the vertices of C in clockwise order by $v_{i_1}, v_{i_2}, \dots, v_{i_n}$ where $\{i_1, i_2, \dots, i_n\}$ is a permutation of $\{1, \dots, n\}$. Then the vertices v_{i_k}, $v_{i_{k+1}}$ are connected by an edge in C for any $1 \leq k \leq n$. That is $[A]_{i_k, i_{k+1}} = 1 \neq 0$.

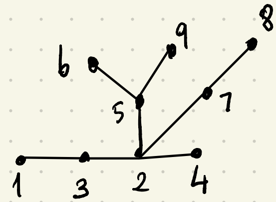
Now starting at v_{i_k}, transverse C in the counterclockwise order to $v_{i_{k+1}}$. This is a walk from v_{i_k} to $v_{i_{k+1}}$ which uses $(n-1)$ edges. Therefore $[A^{n-1}]_{i_k, i_{k+1}} \neq 0$ for any $1 \leq k \leq n$.

4 a)

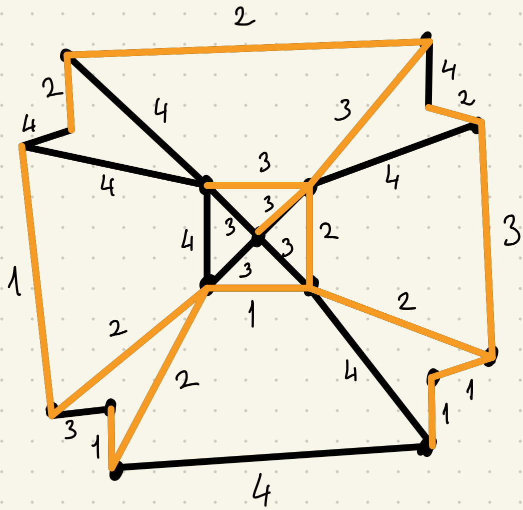


$$\begin{array}{ccccccccc} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \sigma = (& \cancel{3}, & \cancel{2}, & \cancel{2}, & \cancel{5}, & \cancel{7}, & \cancel{2}, & \cancel{5} &) \end{array}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$



b)



$$\text{weight} = 5 \times 1 + 7 \times 2 + 4 \times 3 = 31$$

5 a) Let $c = \sum_{R \text{ region}} b(R)$

where $b(R)$ is the bound degree of R = # of edges on the boundary of R

Since every edge bounds at most two regions:

$$c \leq 2q$$

Also $c \geqslant \underbrace{4(r-1)}_{\text{G has no 3-cycles}} + \underbrace{6}_{\text{has a region of bound degree 6}} = 4r+2$

Then $2q \geqslant 4r+2$ so $q \geqslant 2r+1$ as required.

b) By Euler's formula, $n-q+r=2$ so $r=2-n+q$.

Substituting in (a),

$$q \geqslant 2r+1 = 2(2-n+q)+1 = 5-2n+2q \implies 2n \geqslant 5+q.$$