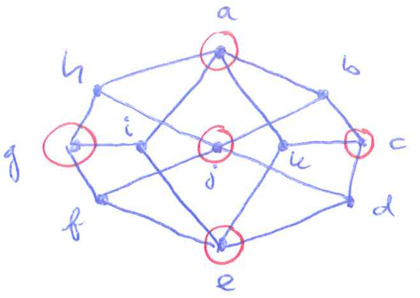
1a) Yes:



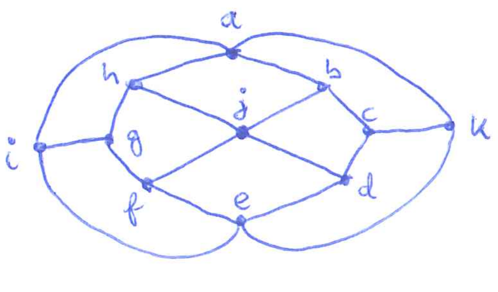
Partite sets:

$X = \{a, c, e, g, j\}$

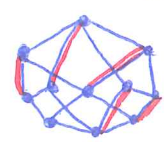
$Y = \{b, d, f, h, i, k\}$

1b) No. G is bipartite with $|X| = 5$ and $|Y| = 6$.
Each edge in G connects a vertex in X with a vertex in Y, so any cycle in G alternately passes through a vertex in X and then a vertex in Y. Suppose w.l.o.g. that the Hamiltonian cycle starts in vertex $y_1 \in Y$. Then suppose there exists a Hamiltonian path $y_1, x_1, y_2, x_2, y_3, x_3, y_4, x_4, y_5, x_5, y_6$, where $x_i \in X$ and $y_i \in Y$. (If this path does not exist, there won't be a Hamiltonian cycle either.) This path uses all vertices from X and Y, but it starts in Y and ends in Y. The vertices in Y are not adjacent, so there is no Hamiltonian cycle.

1c) Yes:



1d) The edges of a matching cannot share end vertices, so each vertex of G can be used at most once. There are 11 vertices, with each edge I use 2, so the maximum number of edges in a matching in G is at most 5. The picture shows that a matching of 5 edges is possible, so the max. number of edges in a matching is 5.



2) Suppose G has order n and less than $n(n-1)/2$ edges. This means that G is not complete.
There are at least two nonadjacent vertices. These two vertices can be given the same colour, and the remaining $n-2$ vertices can be coloured using $n-2$ different colours. Therefore G is $(n-1)$-colourable. ⊠

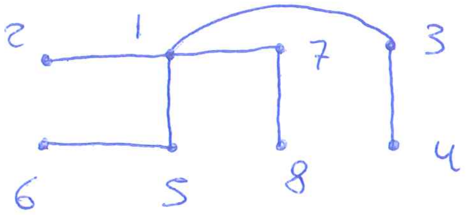
3) ($\Rightarrow$) Suppose G is bipartite and k is odd. The (i,i) entries of A^k give the number of walks from v_i to v_i using k edges. Suppose G has partite sets X and Y. Every step of a walk in a bipartite graph takes you from X to Y or from Y to X, therefore there are no walks of odd length that start and end in the same partite set. So if k is odd, then $[A^k]_{i,i} = 0$ $\forall i$, so $Tr(A^k) = \sum_i [A^k]_{i,i} = 0$.

($\Leftarrow$) Suppose G has adjacency matrix A with the property that $Tr(A^k) = 0$ for all positive odd integers k. An adjacency matrix cannot have negative entries, so this means $[A^k]_{i,i} = 0$ $\forall i$. This means that G does not have closed walks of odd length. Therefore G certainly does not have closed cycles of odd length. So G is bipartite (see theorem 1.3). ⊠

4) Let G be a k-regular Eulerian graph of even order. This means G has an even number of vertices n that all have degree k, and k is even for G is Eulerian. We know that the sum of degrees equals twice the number of edges: $\sum_{i=1}^{n} deg(v_i) = \sum_{i=1}^{n} k = nk = 2|E(G)|$.
Since n and k are even, we can write $n = 2a$ and $k = 2b$ for some $a, b \in \mathbb{N}$. Then $2|E(G)| = 2a \cdot 2b$, so $|E(G)| = 2ab$. So the number of edges is even. ⊠

5) $\sigma = \cancel{1}, \cancel{3}, \cancel{1}, \cancel{5}, \cancel{1}, \cancel{7}$

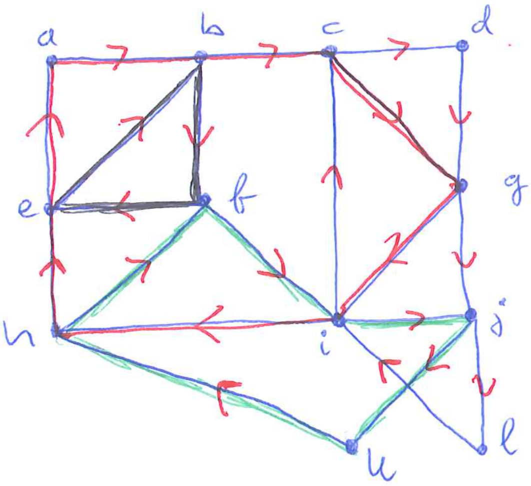
$S = \{\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}8\}$



$$\left.\begin{array}{ll} 2 & 1 \\ 4 & 3 \\ 3 & 1 \\ 6 & 5 \\ 5 & 1 \\ 1 & 7 \end{array}\right\}$$ lowest number in S_i that is not in σ_i with first number in σ_i

7 8 ← two numbers left in S_6

6)



$R_1 = a, b, c, g, i, h, e, a$

$Q_1 = b, f, e, b$

$R_2 = a, b, f, e, b, c, g, i, h, e, a$

$Q_2 = f, i, j, k, h, f$

$R_3 = a, b, f, i, j, k, h, f, e, b, c, g, i, h, e, a$

$Q_3 = i, c, d, g, j, l, i$

<u>An</u> Eulerian circuit is

a, b, f, i, c, d, g, j, l, i, j, k, h, f, e, b, c, g, i, h, e, a

7) G is connected, planar, not a tree, v vertices, e edges, f regions and at least m vertices of degree 1.

a) $b(R)$ is the number of edges that are on the boundary of a region R. Edges that are not on a cycle are not counted, these are counted in $c(R)$, for these edges come only into contact with one region.

There are at least m vertices of degree 1, so there are at least m edges that come into contact with exactly one region.

Edges can be on the boundary of two regions, then we count edges twice, or come only in contact with one region, then we count edges once. Therefore

$$2e = \sum_R b(R) + 2\sum_R c(R) \geq \sum_R b(R) + 2m.$$

G is connected and not a tree, therefore it has at least one cycle. Therefore each region is bounded by at least 3 edges, so $\sum_R b(R) \geq 3f$.

Hence $2e \geq 3f + 2m$. $\boxtimes$

b) We know $2e \geq 3f + 2m$ and Euler's formula $v - e + f = 2$. Combining the two gives

$2e \geq 3(2 - v + e) + 2m$, so $3v - 6 \geq 3e - 2e + 2m$.

Hence $3(v-2) \geq e + 2m$. $\boxtimes$