Practice Final Exam	**Discrete Mathematics**
VU Amsterdam	**2018 (time: 2 hours)**

- Except for question (1): answers without explanation score poorly, so include explanations.
- The use of books, lecture notes, calculators, etc. is *not* allowed.

(1) For both of the following permutations in S_9, write as a product of disjoint cycles:
  (a) (325871)(123456789)(178523)
  (b) (95)(693)(4698)(23847)

(2) Show that for all $n \in \mathbb{Z}_{>0}$

$$\sum_{a+b+c+d=n} \binom{n}{a,b,c,d} 2^{b+d} 3^{c+d} (-1)^d = 0.$$

(3) Compute the number of $\pi \in S_{30}$ that can be written as a 9-cycle times a 10-cycle times an 11-cycle where the cycles are (mutually) disjoint.

(4) Recall that Euler's totient function $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is given by

$$\varphi(n) = |\{k \in \{1, \ldots, n\} \,|\, \gcd(k, n) = 1\}|.$$

Deduce that for all $n \in \mathbb{Z}_{>0}$:

$$\varphi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

(5) The *Pell-Lucas numbers* are recursively defined by

$$Q_0 = Q_1 = 2, \quad Q_k = 2Q_{k-1} + Q_{k-2} \text{ for } k \in \mathbb{Z}_{\geq 2}.$$

Find a formula for Q_k ($k \in \mathbb{Z}_{\geq 0}$) in terms of the *silver ratio* $\delta = 1 + \sqrt{2}$ and $\hat{\delta} = 1 - \sqrt{2}$.

(6) Let $a_0, b \in \mathbb{R}$ and consider the recurrence relation

$$a_k = b a_{k-1} + 2b^k \text{ for } k \in \mathbb{Z}_{>0}.$$

Solve for a_k ($k \in \mathbb{Z}_{\geq 0}$) in terms of a_0 and b.

(7) You are given that the full symmetry group (rotations and reflections) of a regular tetrahedron is S_4. A symmetry corresponds to a permutation $\pi \in S_4$ of the 4 vertices. How many ways are there to color the 4 vertices of a regular tetrahedron using $k \in \mathbb{Z}_{>0}$ different colors if we discount rotations and reflections?
(Each vertex gets exactly one color and different vertices might get the same color.)