Practice Final Exam Discrete Mathematics.

1a) $(3\,2\,5\,8\,7\,1)(1\,2\,3\,4\,5\,6\,7\,8\,9)(1\,7\,8\,5\,2\,3)$

$= (1\,7\,9\,3\,5\,2\,4\,8\,6)$

1b) $(9\,5)(6\,9\,3)(4\,6\,9\,8)(2\,3\,8\,4\,7)$

$= (2\,6\,3\,4\,7)(5\,9\,8)$

2) $\sum_{a+b+c+d=n} \binom{n}{a,b,c,d} 2^{b+d} 3^{c+d} (-1)^d$

$= \sum_{a+b+c+d=n} \binom{n}{a,b,c,d} 1^a 2^b 3^c (-6)^d$

By the multinomial theorem, this is equal to $(1+2+3-6)^n = 0$, $\forall n \in \mathbb{Z}_{>0}$.

3) $\pi \in S_{30}$

write as (9-cycle)(10-cycle) (11-cycle)

There are 30! ways to order the numbers from the set $\{1, 2, \dots, 30\}$, i.e. 30! permutations.

(Then the 9-cycle contains the first nine numbers of a certain permutation, the 10-cycle the next ten and the 11-cycle the last eleven numbers.)

But each cycle can start at any point, for example $(3\,4\,2\,8) = (8\,3\,4\,2) = (2\,8\,3\,4) = (4\,2\,8\,3)$.

Therefore, divide 30! by the cycle lengths:

$$\frac{30!}{9 \cdot 10 \cdot 11}.$$

Another way to think about this:

$\binom{30}{9,10,11}$ ways to choose the numbers for each cycle.

Then there are 8! ways to order the numbers in the 9-cycle, for one number must be fixed (cycle can start at any point). Similarly 9! ways for 10-cycle and 10! for 11-cycle.

So $\binom{30}{9,10,11} \cdot 8!\,9!\,10! = \frac{30!}{9!\,10!\,11!} \cdot 8!\,9!\,10! = \frac{30!\,8!}{11!} = \frac{30!}{9 \cdot 10 \cdot 11}$

4) $\varphi(n)$ counts the number of positive integers up to n that are relatively prime to n.

Use the Principle of Inclusion and Exclusion.

Suppose n is divisible by primes p_i; $p_i \mid n$ for all i.

Let a_i be the property "is divisible by p_i".

Then $N_0 = \varphi(n)$, $N = n$,

$N(a_i) = \frac{n}{p_i}$ (number of integers up to n that are divisible by p_i)

$N(a_i a_j) = \frac{n}{p_i p_j}$, $N(a_i a_j a_k) = \frac{n}{p_i p_j p_k}$, etc.

$$N_0 = N - \sum_i N(a_i) + \sum_{i<j} N(a_i a_j) - \sum_{i<j<k} N(a_i a_j a_k) + \cdots$$

$$\varphi(n) = n - \sum_i \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \sum_{i<j<k} \frac{n}{p_i p_j p_k} + \cdots$$

$$= n\left(1 - \sum_i \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \sum_{i<j<k} \frac{1}{p_i p_j p_k} + \cdots\right)$$

$$= n \prod_i \left(1 - \frac{1}{p_i}\right).$$

5) $Q_0 = 2$, $Q_1 = 2$, $Q_k = 2Q_{k-1} + Q_{k-2}$ for $k \geq 2$.

$$G(x) = \sum_{k\geq 0} Q_k x^k = Q_0 + Q_1 x + \sum_{k\geq 2} (2Q_{k-1} + Q_{k-2}) x^k.$$

$$= 2 + 2x + 2\sum_{k\geq 2} Q_{k-1} x^k + \sum_{k\geq 2} Q_{k-2} x^k$$

$$= 2 + 2x + 2\sum_{k\geq 1} Q_k x^{k+1} + \sum_{k\geq 0} Q_k x^{k+2}$$

$$= 2 + 2x + 2x\sum_{k\geq 0} Q_k x^k - 2Q_0 x + x^2 \sum_{k\geq 0} Q_k x^k$$

$$G(x) = 2 - 2x + 2x\,G(x) + x^2 G(x)$$

$$(1 - 2x - x^2)\,G(x) = 2 - 2x$$

$$G(x) = \frac{2x - 2}{x^2 + 2x - 1}$$

See next page

$$\frac{2x-2}{x^2+2x-1} = \frac{2x-2}{(x+\delta)(x+\hat\delta)} = \frac{A}{x+\delta} + \frac{B}{x+\hat\delta} = \frac{A(x+\hat\delta)+B(x+\delta)}{(x+\delta)(x+\hat\delta)}$$

So $(A+B)x + A\hat\delta + B\delta = 2x - 2$

$$\begin{cases} A + B = 2 \\ A\hat\delta + B\delta = -2 \end{cases} \longrightarrow \begin{cases} A\delta + B\delta = 2\delta \\ A\hat\delta + B\delta = -2 \end{cases} -$$

$$A\delta - A\hat\delta = 2\delta + 2$$

$$A(\delta - \hat\delta) = 2\delta + 2$$

$$A = \frac{2\delta+2}{\delta-\hat\delta} = \frac{2\delta+2}{2\sqrt{2}}$$

$$A = \frac{\delta+1}{\sqrt{2}}$$

$$B = 2 - A = 2 - \frac{\delta+1}{\sqrt{2}} = \frac{2\sqrt{2}-\delta-1}{\sqrt{2}} = \frac{\sqrt{2}-2}{\sqrt{2}} = -\frac{2-\sqrt{2}}{\sqrt{2}}$$

$$B = -\frac{\hat\delta+1}{\sqrt{2}}$$

So
$$G(x) = \frac{1}{\sqrt{2}}\left(\frac{\delta+1}{x+\delta} - \frac{\hat\delta+1}{x+\hat\delta}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1+1/\delta}{x/\delta+1} - \frac{1+1/\hat\delta}{x/\hat\delta+1}\right)$$

note that $1/\delta = -\hat\delta$

$$= \frac{1}{\sqrt{2}}\left(\frac{1-\hat\delta}{1-\hat\delta x} - \frac{1-\delta}{1-\delta x}\right)$$

It follows that

$$G(x) = \frac{1}{\sqrt{2}}\left((1-\hat\delta)\sum_{k\geq 0}(\hat\delta x)^k - (1-\delta)\sum_{k\geq 0}(\delta x)^k\right)$$

$$= \sum_{k\geq 0}\frac{1}{\sqrt{2}}\left((1-\hat\delta)\hat\delta^k - (1-\delta)\delta^k\right)x^k$$

So
$$Q_k = \frac{1}{\sqrt{2}}\left((1-\hat\delta)\hat\delta^k - (1-\delta)\delta^k\right)$$

$$= \frac{1-\hat\delta}{\sqrt{2}}\hat\delta^k - \frac{1-\delta}{\sqrt{2}}\delta^k$$

$$= \hat\delta^k + \delta^k.$$

6) $a_0, b \in \mathbb{R}$, $a_k = ba_{k+1} + 2b^k$ for $k \geq 1$

$$G(x) = \sum_{k\geq 0} a_k x^k = a_0 + \sum_{k\geq 1}(ba_{k-1} + 2b^k) x^k$$

$$= a_0 + b\sum_{k\geq 1} a_{k-1} x^k + 2\sum_{k\geq 1} b^k x^k$$

$$= a_0 + b\sum_{k\geq 0} a_k x^{k+1} + 2\sum_{k\geq 0} b^{k+1} x^{k+1}$$

$$= a_0 + bx\sum_{k\geq 0} a_k x^k + 2bx\sum_{k\geq 0} (bx)^k$$

$$G(x) = a_0 + bx\,G(x) + 2bx \cdot \frac{1}{1-bx}$$

$$(1-bx)\,G(x) = a_0 + \frac{2bx}{1-bx}$$

$$G(x) = \frac{a_0}{1-bx} + \frac{2bx}{(1-bx)^2}$$

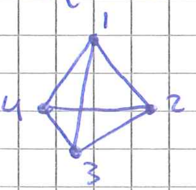
It follows that

$$G(x) = a_0 \sum_{k\geq 0} (bx)^k + 2\sum_{k\geq 0} k\,(bx)^k$$

$$= \sum_{k\geq 0} \left(a_0 b^k + 2k\,b^k\right) x^k .$$

So $a_k = a_0 b^k + 2k b^k = (a_0 + 2k)\,b^k$.

7) S_4 is the full symmetry group of a regular tetrahedron.



Burnside's lemma:

$$N = \frac{1}{|G|}\sum_{\pi\in G} |C_\pi| .$$

Here $G = S_4$, and $|G| = 4! = 24$.

$|C_{(1)}| = K^4$, all colourings are invariant under identity.

$|C_{(1234)}| = K$, only invariant if all vertices have same colour. There are $3! = 6$ cycles of length $4 \to 6K$.

$|C_{(123)}| = K^2$, at least three vertices must have the same colour. There are $2!\cdot 4 = 8$ cycles of length $3 \to 8K^2$

$|C_{(12)}| = K^3$, at least two vertices must have the same colour. There are $4\cdot 3/2 = 6$ cycles of length $2 \to 6K^3$.

$|C_{(12)(34)}| = K^2$, at least two pairs must have the same colour. There are three different cycles $(12)(34)$ and $(13)(24)$ and $(14)(23) \to 3K^2$.

So $N = \frac{1}{24}\left(K^4 + 6K^3 + 11K^2 + 6K\right)$.