Retake exam Differentiëren en Integreren 3

11 June 2015

- This exam consists of **6** questions on **2** pages.
- The total number of points is **45**.
- Grade = $\frac{\text{Total}}{5} + 1$.
- You have **120 minutes** to write the exam.
- Always motivate your answers and write clearly. Please write all answers in English.
- Books, calculators, laptops, smartphones, etc. are not allowed.

1. (5 points) Evaluate the double integral

$$\int_0^2 dy \int_y^2 e^{-x^2}\, dx.$$

2. (3+5 points) Let the function ϕ given by

$$\phi(x, y, z) = \frac{x^3}{3} + x^2 z^2 - 4xyz,$$

and the curve $\mathcal{C}$ parametrized by $\mathbf{r}(t) = (t, \mathrm{e}^{-t}, \mathrm{e}^{t})$, for $t \in [0, 1]$.

(a) Find $\mathbf{G} = \nabla\phi$. Is $\mathbf{G}$ solenoidal?

Consider the vector field $\mathbf{F}$ in $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x^2 + 2xz^2)\,\mathbf{i} - 4xz\,\mathbf{j} + (2x^2 z - 4xy)\,\mathbf{k},$$

(b) Find $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$. [*Hint*: use (a)]

3. (7 points) Use the transformation $u(x, y) = x^2 + y^2$, $v(x, y) = \dfrac{y}{x}$ to evaluate the double integral

$$\iint_D \frac{y^2 + x^2}{x^2}\, dA,$$

where D is the region in the first quadrant under the line $y = 2x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$.

Questions 4–6 are on the next page.

4. (5 points) Evaluate the line integral

$$\oint_{\mathcal{C}} xy^3\,dx + 4x\,dy,$$

where $\mathcal{C}$ is the boundary of the rectangle $R = [2,3] \times [0,1]$ oriented clockwise.

5. (5+8 points) Consider the region D bounded by the surface $z = x^2 + y^2$ (top), the xy-plane (bottom), and the four planes $x = -1$, $x = 1$, $y = -1$, $y = 1$ (side). Denote the boundary of D by $\mathcal{S}$. Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be the top, bottom, and side part of $\mathcal{S}$, with oriented boundaries $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Let $\mathbf{F}(x,y,z) = z\,\mathbf{i} - y\,\mathbf{j} + x^2\,\mathbf{k}$.

(a) Consider two vector fields $\mathbf{F}$ and $\operatorname{curl}\mathbf{F}$. The divergence theorem and Stokes's theorem allow to express flux integrals of these vector fields across certain surfaces as triple or line integrals. Choose $\mathcal{S}$ or $\mathcal{S}_3$ for each of the vector fields and state the assertion of the corresponding theorem. Motivate your answer. [Note: You do not have to evaluate the integrals.]

(b) Find the upward flux of $\mathbf{F}$ across $\mathcal{S}_1$ directly, i.e., without applying any of the three theorems of vector calculus.

6. (7 points) Evaluate the surface integral

$$\iint_{\mathcal{S}} \frac{z + 2y^2}{\sqrt{1 + 4x^2 + 4y^2}}\,dS,$$

where $\mathcal{S}$ is the part of the surface $x^2 - y^2 - z = 4$ that lies inside the cylinder $x^2 + y^2 = 1$.

Good luck!