Final exam Differentiëren en Integreren 3

25 March 2015

- This exam consists of **5** questions.
- The total number of points is **36**.
- Grade = $\frac{\text{Total}}{4} + 1$.
- You have **120 minutes** to write the exam.
- Always motivate your answers and write clearly. Please write all answers in English.
- Books, calculators, laptops, smartphones, etc. are not allowed.

1. (4 points) Evaluate the line integral

$$\oint_{\mathcal{C}} \left(2x^2y - \sin x^2\right) dx + \left(x^3 + xy^2 + \tan y^3\right) dy,$$

where $\mathcal{C}$ is the boundary of the half-disk $x^2 + y^2 \leq 1$, $y \geq 0$ oriented counterclockwise.

2. (8 points) Consider two vector fields in $\mathbb{R}^3$

$$\mathbf{F}(x, y, z) = (2xy + z)\,\mathbf{i} + (x^2 + 3y^2z^2)\,\mathbf{j} + (2zy^3 + x)\,\mathbf{k},$$
$$\mathbf{G}(x, y, z) = (xy^2z + x^2 \sin z)\,\mathbf{i} - \frac{y^3z}{3}\,\mathbf{j} + (2x\cos z - y)\,\mathbf{k}.$$

Show that $\mathbf{F}$ is irrotational and $\mathbf{G}$ is solenoidal.

3. (6 points) Use Stokes's theorem to find the circulation of the vector field

$$\mathbf{F}(x, y, z) = (y^2 + 2xy - 3y)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j} + \sin z\mathbf{k}$$

around the curve $\mathcal{C}$ of intersection of the surfaces $z = 2y^2 + 1$, $x = 2$, $x = 3$, $y = -2$, and $y = 3$, oriented counterclockwise as seen from a point high on the z-axis.

4. (6 points) Find the area of the surface $\mathcal{S}$ parametrized by

$$\mathbf{r}(u, v) = 2u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + (4 - u\cos v - u\sin v)\,\mathbf{k}, \qquad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

5. (6+6 points) Let $\mathcal{S}$ be the boundary of the region bounded by $z = 4 - x^2 - y^2$ and $z = 0$. Consider the vector field

$$\mathbf{F}(x, y, z) = (xz\sin(yz) + x^3)\,\mathbf{i} + \cos(yz)\,\mathbf{j} + (3zy^2 - 1)\,\mathbf{k}.$$

(a) Find the outward flux

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

(b) Let $\mathcal{S}_1$ be the piece of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane, i.e. the upper part of the surface $\mathcal{S}$. Use above result to find the upward flux across the surface $\mathcal{S}_1$.