## FINAL EXAM FOR DIFFERENTIAL GEOMETRY, FALL 2014

Time: 15:15-18:00 - Books, notes, calculator, etc. are not permitted!

Use for each of the 3 exercises a separate piece of paper!

Do not forget to write your name and student number (UvA/VU) on all papers!

Grading: your grade = 1/3 times your points

## Exercise 1

- a) How is the differential of a smooth map  $f: M \to N$  at a point  $p \in M$  defined? When is f called an immersion? And is the image  $f(M) \subset N$  of an immersion f a submanifold of N? (3P)
- b) Let  $\mathbb{R}$  be the real line with its usual manifold structure. Let M denote the manifold which equals  $\mathbb{R}$  as a set but with the manifold structure given by the coordinate chart  $\phi: \mathbb{R} \to M, \ x \mapsto x^3$ . Show that the identity map  $\mathbb{R} \to M$  is a homeomorphism, but not a diffeomorphism. Are M and  $\mathbb{R}$  diffeomorphic? (3P)
- c) Show that  $SL(2) := \{A \in \mathbb{R}^{2 \times 2} : \det(A) = 1\}$  is a submanifold of  $\mathbb{R}^4$ . What is its dimension ? (3P)

## Exercise 2

- a) What is a Riemannian metric on a smooth manifold M? Does there always exist a Riemannian metric on any submanifold of  $\mathbb{R}^n$ ? (3P)
- b) Give two ways how to define the topology of a manifold M equipped with a Riemannian metric g. (3P)
- c) Compute the Lie bracket of the vector fields  $X(x) = (-x_2, x_1, 0)$  and  $Y(x) = (x_1x_3, x_2x_3, -x_1^2 x_2^2)$  on  $S^2 \subset \mathbb{R}^3$ , where  $(x_1, x_2, x_3)$  are the coordinates on  $\mathbb{R}^3$ . What can be said about the flows of X and Y? (3P)
- d) Let  $\mathfrak{g}$  be the Lie algebra of a Lie group G and for each  $\xi \in \mathfrak{g}$  let  $\phi_t^{\xi}$  denote the flow of the corresponding left-invariant vector field. Show that the map  $\exp : \mathfrak{g} \to G$ ,  $\xi \mapsto \phi_1^{\xi}(e)$  maps an open neighborhood of 0 in  $\mathfrak{g}$  diffeomorphically to an open neighborhood of the neutral element e in G. (3P)

## Exercise 3

- a) Explain the relation between the exterior derivative and divergence and curl of a vector field on  $M = \mathbb{R}^3$ . (3P)
- b) Compute step by step how a two-form  $\omega = f(x_1, x_2) dx_1 \wedge dx_2$  on  $\mathbb{R}^2$  changes under a coordinate transformation  $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ ,  $(y_1, y_2) \mapsto (x_1, x_2)$ . Explain the application to integration on (two-dimensional) manifolds. (3P)
- c) Show that the one-form  $\omega = (x_1 dx_2 x_2 dx_1)/(x_1^2 + x_2^2)$  on  $\mathbb{R}^2 \setminus \{0\}$  is closed, but not exact. (3P)