VU UNIVERSITY AMSTERDAM

Faculty of Economics and Business Administration

# Derivatives 4.4

## – How to Prepare for and handle the Exam –

Norman Seeger

*How to prepare for the exam:*

- Main material to prepare for the exam are the lecture slides. Use the Hull book to read up on things which are not clear to you in the slides.
- Topics in the slides I have been repeatedly discussing and gave more attention during class are more important to me and, therefore, are also more important for the exam.
- The VBA tutorial problems are equally important for the exam. Maker sure you can solve all assignment questions on your own. Also understand your own VBA solutions.
- VBA tutorial problems are designed in a way to also strengthen your theoretical understanding of the material. So working on those is a win-win :-) .
- Solve the additional exercises provided to you below.
- Make your own exercises. Think from my point of view what might be interesting to ask you.

*Structure of exam:*

- Provide your name and student id at the beginning of the exam
- The exam will consist of a majority open questions for which the answers need to be typed in word.
- The solutions have to be written down in the space indicated by the boxes. Write only in the boxes! The provided space gives you and indication for how much space you need to write down the solution I expect from you.
- There will be also some multiple choice questions for which the answers need to be filled in in an excel sheet.

- There will be a good part of programming problems (roughly 1/3 of the exam) that need to be solved by programming VBA. The numerical results of the programs need to be transferred to an excel sheet and will be graded.
- With all questions there comes an indication of how many points you can earn for the total question and how much each subquestion is worth. You can interpret the points also in way of how much time to spend for solving a question. One point is one minute of time. If one subquestion is worth 5 points you should spend roughly 5 minutes to solve the question.
- At the end of the exam there is a formulary. Check out what is provide to you. The formulary is not provided to you before the exam. If a big formula is needed to solve a question I will provide it in the formulary. e.g. Black-Scholes formula. Short standard formulas such as $q$ probabilities in the CRR model you have to know by heart.
- Use of a graphical calculator is not allowed. You have excel at your disposal during the exam; the best graphical calculator of the world.

*Handling the exam:*

- Scan the exam first, to check which are the questions that are most easy to solve for you. Start with those in order to collect as much points as possible
- Read the questions carefully; think for a moment about what is the best way to solve the question. Often there are multiple ways of solving a question but one way might be smarter than another.
- If you cannot solve a subquestion go on to the next one and come back later when you solved all the rest. Collect points!
- Sometimes there is more information than needed

Faculty of Economics and Business Administration

Derivatives 4.4

– Additional Exercises –

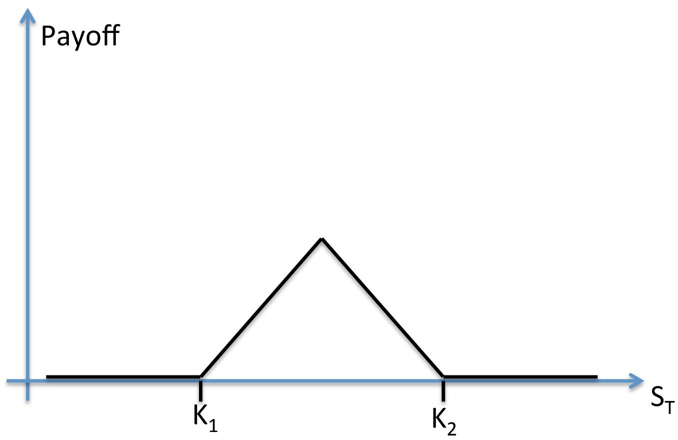
Norman Seeger

# Part I

# Old Exam Exercises

**1. Structuring and Replication** **(10 Points)**

Consider the following payoff profile.




(a) Give one portfolio of securities that replicates this payoff profile. For your replicating portfolio you are allowed to choose out of the following securities. All derivatives are written on the same underlying $S$ and have the same time to maturity $T$. One can invest in long and short positions. (6 points)

- Underliyng $S$; $S_0 = 100$
- Bond (notional $\frac{K_1+K_2}{2}$); $B_0 = 90.91$
- Forward($K_1$); $F_0(K_1) = 0$
- Call($K_1$); $C_0(K_1) = 1.20$
- Call($K_2$); $C_0(K_2) = 0.02$
- Call($\frac{K_1+K_2}{2}$); $C_0(\frac{K_1+K_2}{2}) = 0.34$
- Put($K_1$); $P_0(K_1) = 0.01$
- Put($K_2$); $P_0(K_2) = 0.82$
- Put($\frac{K_1+K_2}{2}$); $P_0(\frac{K_1+K_2}{2}) = 0.14$

Replication portfolio: long Call($K_1$) + 2 short Call($\frac{K_1+K_2}{2}$) + long Call($K_2$)

(b) What will be the price of the replicating portfolio of the above given payoff profile. (4 points)

price of replication portfolio $= 1.2 - 2 \cdot 0.34 + 0.02 = 0.54$

price of replication portfolio $= 0.54$

**2. Forward Contract** **(10 Points)**

Assume that the current ($t = 0$) stock price equals 40 and the stock pays no dividends. The risk-free interest rate equals 10% (continuous compounding). Consider a forward contract on this stock that expires in one year ($t = 1$).

(a) Determine the forward price, $F_0$, and the forward value, $FV_0$, at time $t = 0$. (4 points)

- Forward price

$$F_0 = S_0 e^{rT} = 40e^{0.1\cdot 1} = 44.2068$$

- Forward value

$$FV = 0$$

- Forward price

$$F_0 = 44.2068$$

- Forward value

$$FV = 0$$

(b) During the next six months the stock price rises to 45. Determine the forward price, $F_{0.5}$, and the forward value, $FV_{0.5}$, in six months, i.e. at time $t = 0.5$. (6 points)

- Forward price

$$F_{0.5} = S_t e^{r(T-t)} = 45e^{0.1\cdot 0.5} = 47.3072$$

- Forward value

$$FV = (F_{0.5} - F_0)e^{-r(T-t)} = (47.3072 - 44.2068)e^{-0.1\cdot 0.5} = 2.9492$$

- Forward price

$$F_{0.5} = 47.3072$$

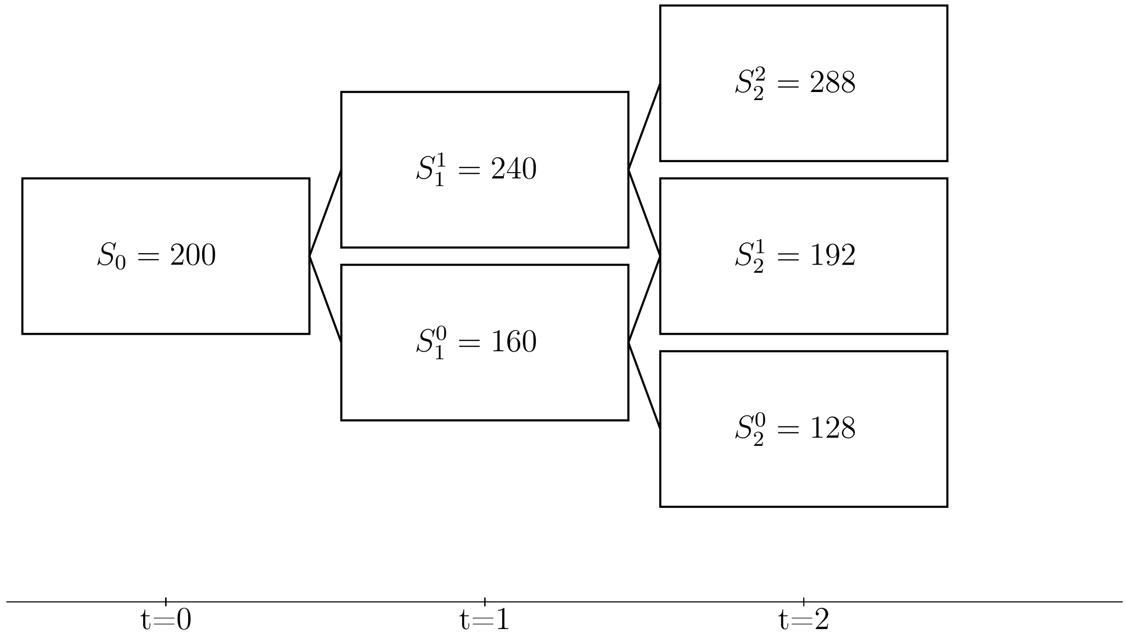
- Forward value

$$FV = 2.9492$$

**3. Binomial Model** **(30 Points)**

Consider the following two-period binomial tree for a non-dividend paying stock. The length of one period is one year. The risk-free rate is 5% (continuous compounding) and the current stock price is $S_0 = 200$.

*Round all your factors to three decimal places.*




(a) Determine the price of a European put with exercise price $K = 200$ that expires in two years ($t = 2$) using risk-neutral valuation. (8 points)

Calculate the risk neutral probability and the payoff at maturity (see tree): $q = \frac{e^{0.05}-0.8}{1.2-0.8} = 0.628$. Proceed backwards in the tree:

$$P_0 = e^{-0.05\cdot 2}[2q(1-q)\cdot 8 + (1-q)^2 \cdot 72] = 12.398$$

$$P_0 = 12.398$$

(b) Assume that the market price of the European put option in (a) is 5. Which strategy would you use to lock in an arbitrage profit? Give the exact positions and the resulting payments today, in one year, and at maturity. (12 points)

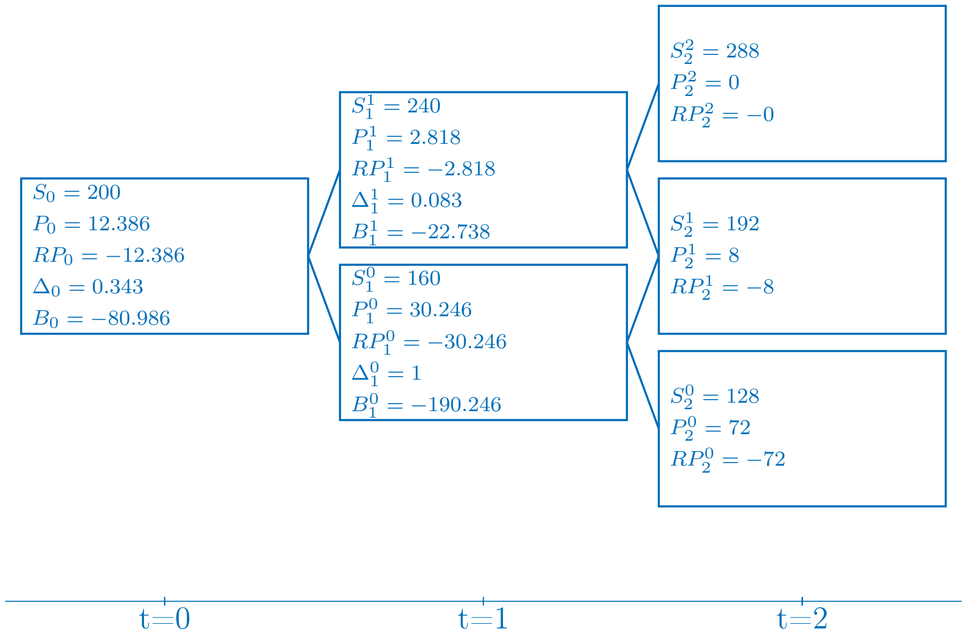
Buy low (the option) sell high (replicating portfolio): Use the value developments!

$$\Delta_{t-1} = \frac{c_t^u - c_t^d}{S_{t-1}(u-d)}$$

$$B_{t-1} = \frac{c_t^u - \Delta_{t-1} S_{t-1} u}{e^{r\Delta t}}$$

- That is the replicating strategy consists of buying one put (long in put) and selling the replicating portfolio of the put (short replicating portfolio). The long position in the put and the short position in the replicating portfolio have perfectly offsetting cash flows in all states in the futures.

- Positions in stock and bond: in general, being short in an European put means being long in stock and being short in bond
- Payment today: $12.40 - 5 = 7.40$
- Payments in $t = 1$ and $t = 2$: 0




Note: Rounding errors due to taking just 3 decimals.

(c) Calculate the value of a European call with the same characteristics as the put option in (a) using put-call-parity. (2 points)

$$C_0 = 12.398 + 200 - 200e^{-0.05 \cdot 2} = 31.431$$

$$C_0 = 31.431$$

(d) Describe the major similarities and differences between the CRR binomial model and the Black and Scholes model by filling in the following table. (8 points)

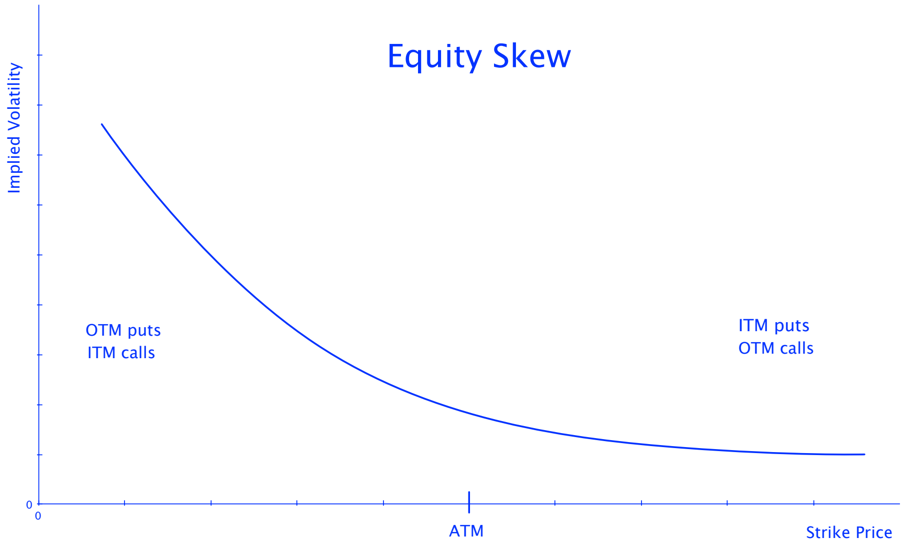
| | Discrete time Binomial Model | Continuous time Model |
|---|---|---|
| uncertainty | | |
| description stock price | | |
| model | | |
| pricing derivatives | | |

| | Discrete time Binomial Model | Continuous time Model |
|---|---|---|
| uncertainty | binomial model with up- and down-moves | Wiener process |
| description of stock price | give stock price in every node of the tree | stochastic differential equation |
| model | Cox/Ross/Rubinstein $S_{t+1} = \begin{cases} S_t u & \text{up} \\ S_t d & \text{down} \end{cases}$ | Black-Scholes<br>Geometric Brownian motion |
| pricing derivatives | replication<br>risk-neutral pricing | replication<br>risk-neutral pricing |

**4. Risk-Neutral Distribution and Implied Volatility Smile (21 Points)**

(a) Draw the implied volatility curve (implied volatility as a function of the strike price) that is typically observed for equity index options. Label your axis.

(6 points)




(b) Describe how the Heston model improves on the Black and Scholes model in order to model underlying security processes more consistently with what is observed in real market data. How is the Heston model able to replicate implied volatility smiles and implied volatility skews. (8 points)

- Heston: stochastic volatility that changes over time and can be correlated with the stock price
- if volatility is uncorrelated with the stock price
  - increases the probability for very low and very high terminal stock prices
  - symmetric smile
- if volatility is negatively correlated with the stock price
  - increases the probability for very low terminal stock prices
  - decreases the probability for very high terminal stock prices
  - decreasing implied volatility, i.e. skew

(c) Explain for pricing which type of options the Heston model is particularly suited. (3 points)

Long time to maturity options

(d) What does the implied volatility smile in (a) imply for the shape of the risk-neutral distribution compared to the (log)normal distribution? Briefly explain the intuition. (4 points)

Distribution is left skewed and leptokurtic.

**5. Greeks and Hedging** **(23 Points)**

(a) Which problem might arise with a delta hedge if a large price movement in the underlying asset occurs? Make a suggestion how to solve that problem. (5 points)

- After one time step a large movement in the underlying can lead to a large difference between the value of the hedge portfolio and the option to be hedged. This implies a large hedging error.
- Delta-gamma hedge

(b) What is the delta of a European call in the Black-Scholes model? What are the bounds for this delta? (3 points)

$\Delta_c = N(d_1), \qquad 0 \leq \Delta_C \leq 1, \qquad N(d_1)$ : Value of the std. normal distribution

(c) Use put-call parity to determine the delta of a European put and bounds for the put delta. (3 points)

Put-Call-Parity: $C + Ke^{-rT} = p + S$

Derivation with respect to S:

$$\Delta_C + 0 = \Delta_p + 1 \quad \rightarrow \quad \Delta_p = N(d_1) - 1 \quad \rightarrow \quad -1 \leq \Delta_p \leq 0$$

(d) Use put-call parity to infer the relationship between the gamma of a European call and a European put option. (4 points)

Put-Call-Parity: $P + S = C + Ke^{-r(T-t)}$
Gamma is determined as a the second derivative with respect to $S$:

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Gamma of a European call equals the gamma of a European put.

(e) There exists a target option, an instrument option, an underlying stock, and a money market account with a risk-free rate of 5%. Construct a delta-gamma-neutral portfolio for hedging a long position in one target option given the information in the following table. (8 points)

| Security | Delta | Gamma |
|---|---|---|
| Target Option (K=50, T=1) | 0.85 | 0.035 |
| Instrument Option (K=70, T=1.5) | 0.7 | 0.04 |

- Delta-neutral: $1 * 0.85 + n_1 * 1 + n_2 * 0.7 = 0$
- Gamma-netural: $1 * 0.035 + n_2 * 0.04 = 0$
- $\Rightarrow n_2 = -0.875$ and $n_1 = -0.2375$


- $\Rightarrow n_2 = -0.875$ and $n_1 = -0.2375$


**6. Model Estimation** **(12 Points)**

(a) Name two methods to calculate model parameters? Explain for each method when it is applied and how the parameters are calculated. (8 points)

i. Estimating model parameters with econometric estimation procedure like ML. Time series of return data is used to estimate parameters, resulting in model parameters under the P measure.

ii. Calculating model parameters by calibrating model to a cross-section of option prices.
Using objective functions to minimize sum of differences between market and model prices, resulting in parameter estimates under the Q measure.

(b) When using "percentage root mean squared error" as loss function for calibration of option pricing models, are OTM options assigned more, less, or equally weight than ITM options? Explain your answer in one to two sentences. (4 points)

$$\%RMSE(\Theta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\left\{C_i^M - C_i(\Theta)\right\}/C_i^M\right]^2}$$

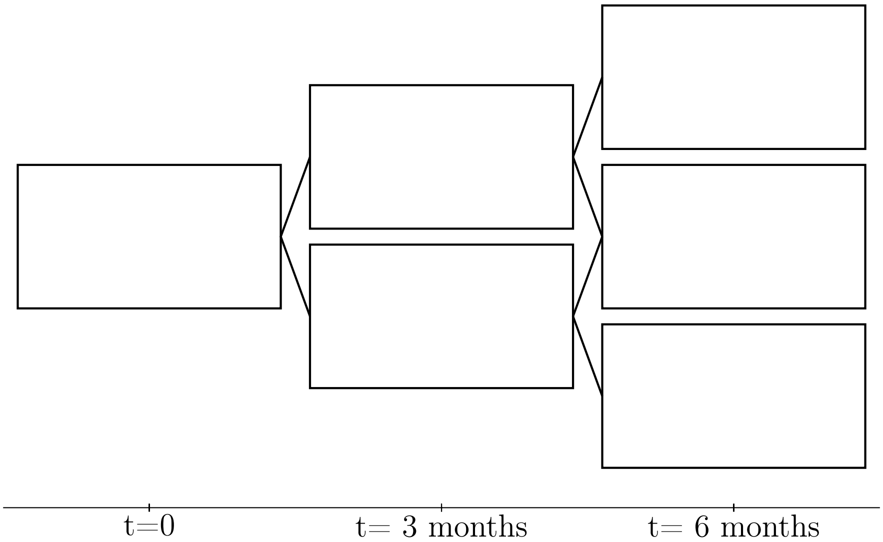
OTM options are assigned more value compared to ITM options. In general OTM options have very small values. This implies that for an OTM option $C_i$ the ratio $\{C_i^M - C_i(\Theta)\}/C_i^M$ is large and increases the %RMSE substantial. An optimization algorithm that minimizes the %RMSE will, therefore, focus on decreasing the ratio $\{C_i^M - C_i(\Theta)\}/C_i^M$ for OTM options; in other words, will concentrate on fitting the OTM options to the market data first.

7. **Binomial Model and Exotic Options** **(24 Points)**

(a) Build a stock price tree based on the CRR model by means of the following parameters: (8 points)

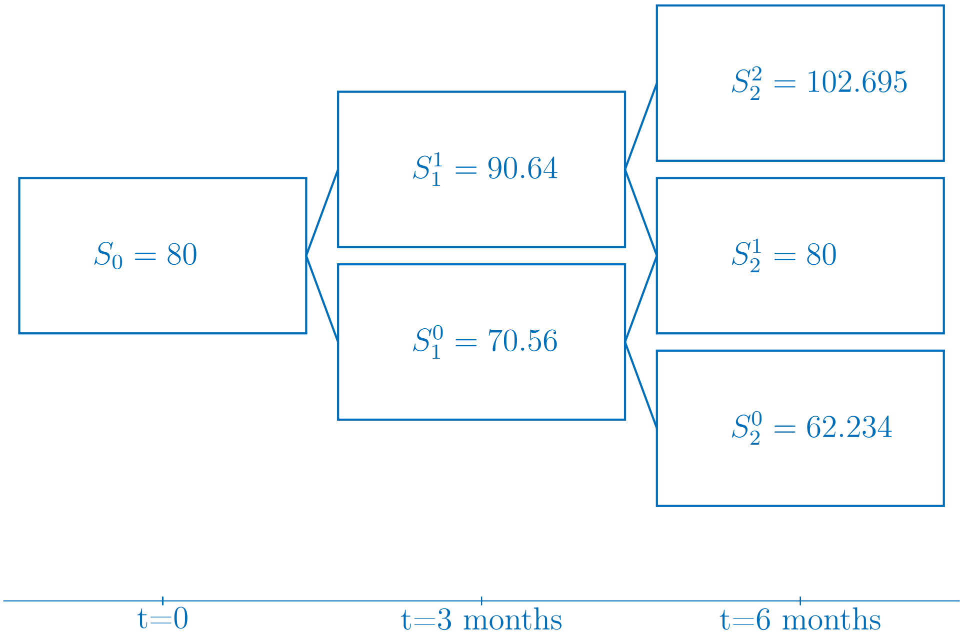
- annual volatility of the stock: 0.25
- annual, continuously compounded risk-free rate: 10%
- stock price today: 80

*Round all your factors to three decimal places.*



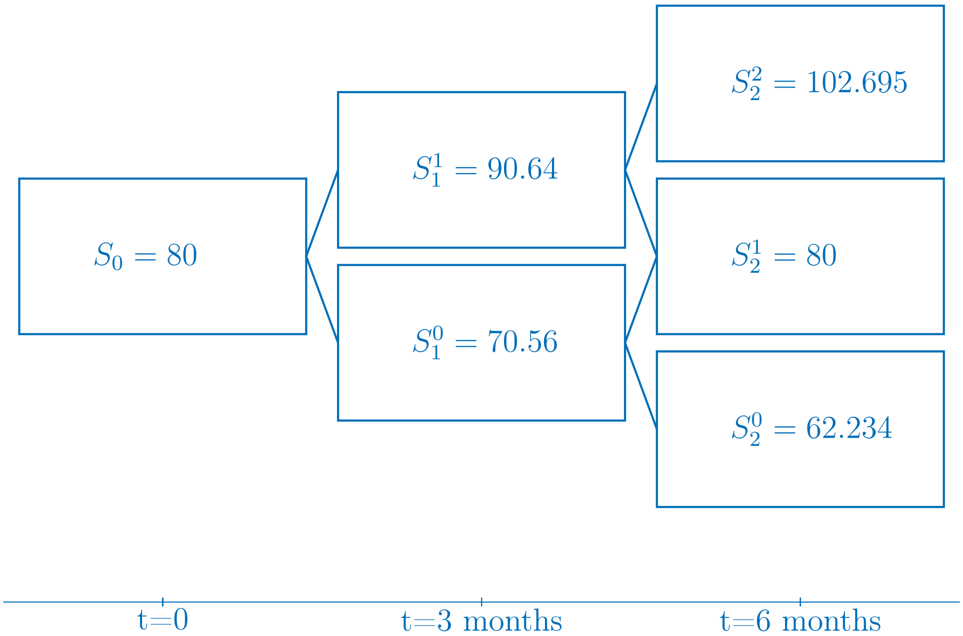

$$u = e^{\sigma\sqrt{\Delta t}} = 1.133$$

$$d = 1/u = 0.882$$




$$u = 1.133$$

$$d = 0.882$$




(b) Determine the price of a European Lookback call that expires in 6 month ($t =$ 6 months) using risk-neutral valuation. (12 points)

- Calculate the risk neutral probability and the payoff at maturity:
  $q = \frac{e^{r\Delta t} - d}{u - d} = 0.571.$
- Determine payoffs of Lookback call at maturity:
  $C_T = \max[S_T - \min_{0 \leq t \leq T}(S_t); 0]$
  $C_2^2(u,u) = 102.695 - 80 = 22.695$
  $C_2^1(u,d) = 80 - 80 = 0$
  $C_2^1(d,u) = 80 - 70.56 = 9.44$
  $C_2^0(d,d) = 62.234 - 62.234 = 0$
- Risk-neutral valuation of Lookback call:
  $C_0 = (22.695 * q^2 + 9.44q(1-q)) \cdot e^{(-r \cdot 6/12)}$
  $C_0 = (22.695 * 0.571^2 + 9.44 * 0.571 * (1 - 0.571)) * e^{-0.1*(6/12)}) = 9.283$

- $C_0 = 9.283$

(c) How does the value of a Lookback call option change as we increase the frequency with which we observe the asset price? (4 points)

As we increase the frequency of observing the asset price, the maximum is equal or lower but certainly not higher. Therefore the value of the option increases.

# Part II

# Additional Exercises

1. **Option Strategies**

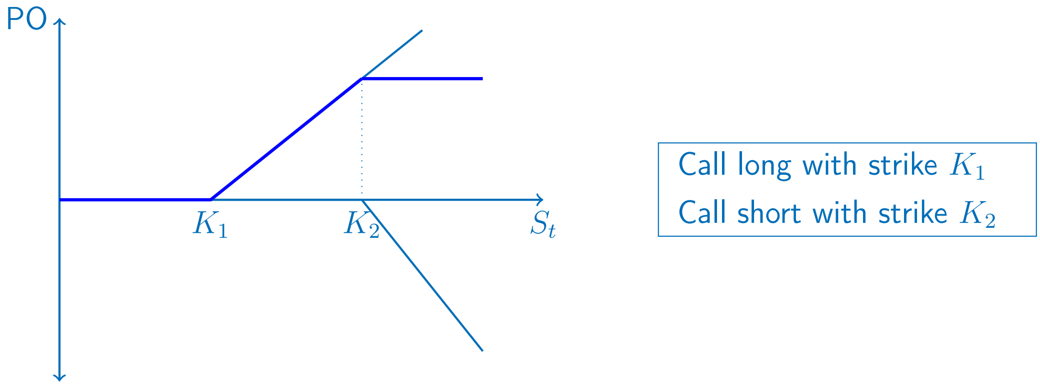
Assume that you purchase a European call option with strike price $K_1$ and sell a call option with strike price $K_2$ with equal time to maturity. ($K_1 < K_2$.)

(a) Draw the payoff diagram of your portfolio at maturity. How is this strategy called?

(b) Assume that the stock price equals $K_1$ today. When would you use this strategy?

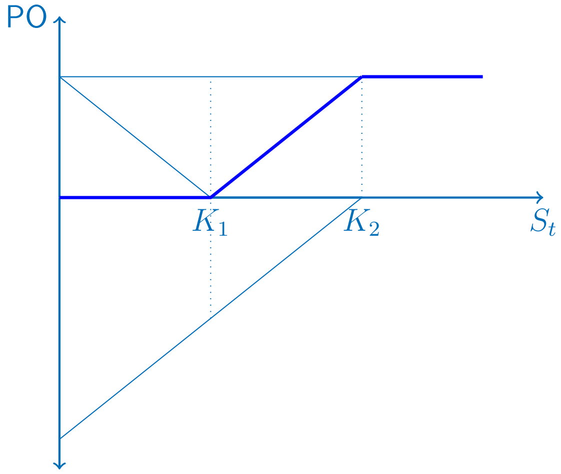
(c) Assume that only European put options are traded in the market. How can you achieve the same payoff structure? Solve this problem graphically as well as computationally.

a) Strategy is called Bull Spread




b) Can be used if you speculate on a moderate increase of stock prices. Your payoff is capped at the level of $K_2 - K_1$, but this is combined with a low initial price for this strategy (note that you sell a call with strike $K_2$ which gives you money).
If you do not expect the stock price to increase beyond $K_2$, you can earn the profit from selling the call with strike $K_2$. And if, additionally, you do not expect the stock price to decrease below $K_1$, you can earn the profit out of the long call with strike $K_1$.

c) put long $K_1$, put short $K_2$, money-market account (risk-free investment) $(K_2 - K_1)e^{-rT}$.
Important: don't forget the MMA, otherwise you will not get all points in the exam! And don't forget to label the axes!




Put long with strike $K_1$
Put short with strike $K_2$
Money-market account $(K_2 - K_1)e^{-rT}$

Computational solution:
Start with the payoff of the call portfolio $(\max[S_T - K_1, 0] - \max[S_T - K_2, 0])$ and compute:

$$\begin{aligned}
& \max(S_T - K_1, 0) - \max(S_T - K_2, 0) \\
= & \max(S_T - K_1, 0) - \max(S_T - K_2, 0) + K_1 - K_2 + K_2 - K_1 + S_T - S_T \\
= & \Big[\max(S_T - K_1, 0) + K_1 - S_T\Big] - \Big[\max(S_T - K_2, 0) + K_2 - S_T\Big] + K_2 - K_1 \\
= & \max(0, K_1 - S_T) - \max(0, K_2 - S_T) + K_2 - K_1
\end{aligned}$$

which is the payoff of the portfolio of puts and money market investment described above.

2. **Forwards and Arbitrage**

The forward price of wheat for delivery in three months is \$3.90 per bushel, while the spot price is \$3.60. The three-month interest rate in continuously compounded terms is 8% per annum. Is there an arbitrage opportunity in this market if wheat may be stored costlessly? If there is an arbitrage opportunity present a way of setting up a strategy to explore the arbitrage opportunity.

The spot price of wheat is \$3.60. Since there are no storage costs, we compute the theoretical forward price of wheat as $3.60 \exp(0.08 \times 3/12) = 3.6727$ which is less than the forward price. Hence, there is an arbitrage opportunity.

In order to arbitrage this situation, we would undertake the following strategy:

- Sell wheat forward at 3.90.
- Buy wheat spot at 3.60.
- Borrow 3.60 for three months .

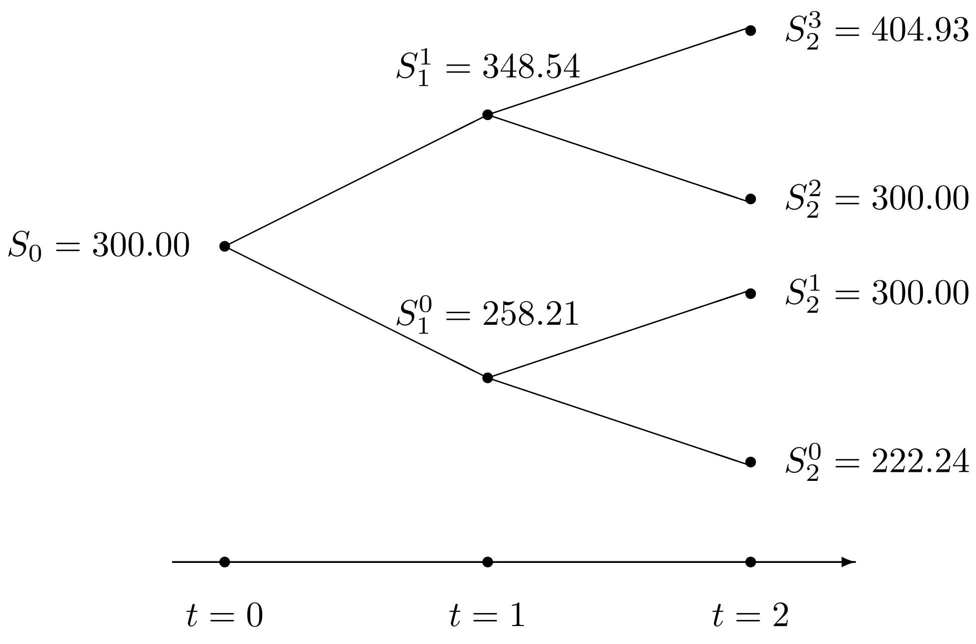
At inception, the net cash-flow is zero. At maturity, we deliver the wheat we own and receive the forward price of \$3.90. We return the borrowed amount with interest for a cash outflow of $3.60 \exp(0.08 \times 3/12) = 3.6727$. This results in a net cash inflow of 0.2273. The following table summarizes:

| | Cash Flows | |
|---|---|---|
| Source | Initial | Final |
| Short Forward | – | +3.9000 |
| Long Spot | -3.6000 | – |
| Borrowing | +3.60000 | -3.6727 |
| Net | – | +0.2273 |

Note that it makes no difference if the contract is cash-settled instead of settled by physical delivery. If it is cash-settled, letting $F_T$ denote the spot price of wheat at date $T$, we receive 3.90 - $F_T$ on the forward contract, sell the spot wheat we own for $F_T$, and repay the borrowing, for exactly the same final cash flow.

3. **Binomial Tree Pricing**

Consider a non-dividend paying stock with current price 300. The development of the stock price is described via the two-period CRR binomial tree given below where the length of one period is one year (i.e. $\Delta t = 1$). The risk-free interest rate is 7% p.a. (continuous compounding).




(a) Calculate the volatility of the stock price used to construct the binomial tree.
*Hint: the tree was constructed using the same volatility factor $\sigma$ in all states and at every point in time.*

$$
\begin{aligned}
u &= S_1^1/S_0 = 1.1618 \\
u &= e^{\sigma\sqrt{\Delta t}} \\
u &= e^{\sigma\cdot 1} \\
\Rightarrow \sigma &= \ln u = \ln(1.1618) \\
\sigma &= 15\%
\end{aligned}
$$

(b) Use replication to determine the price of a European put with exercise price $K = 300$ that expires in one year (i.e. at $\mathbf{t = 1}$).
How many stocks do you have to include in the replicating portfolio at the initial point in time?

Payoff at $t = 1$: $P_1^0 = 41.79$, $P_1^1 = 0$
Replicating portfolio:

$$
\begin{aligned}
\Delta S_1^1 + B\cdot e^{r\Delta t} &= 0 \\
\Delta S_1^0 + B\cdot e^{r\Delta t} &= 41.79 \\
\Rightarrow \Delta &= -0.4626 \\
\Rightarrow B &= 150.3341
\end{aligned}
$$

OR

$$
\begin{aligned}
\Delta &= \frac{P_1^1 - P_1^0}{S_1^1 - S_1^0} = -0.4626 \\
B &= e^{-r\Delta t}(P_1^1 - \Delta\cdot S_1^1) = 150.3341
\end{aligned}
$$

Put price:

$$P_0 = \Delta S_0 + B = 11.5541$$

Stock position: sell $0.4626$ stocks

(c) Calculate the risk-neutral probabilities.

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.07} - 0.8607}{1.1618 - 0.8907} = 0.7034$$

(d) Use risk-neutral pricing to determine the price of a European up-and-out call option with exercise price $K = 290$ and barrier 310, which matures in two years (i.e. at **t=2**). *Hint: If you have not solved part (3c), use a risk-neutral probability of q=0.7.*

Payoffs at $t = 2$: $C_2^0 = 0,\ C_2^1 = 10.00,\ C_2^3 = 0,\ C_2^4 = 0$
Price of $C^{UaO}(K = 290, B = 310)$

$$\begin{aligned} C^{UaO}(K = 290, B = 310) &= e^{-0.07 \cdot 2}(10 \cdot 0.7034 \cdot (1 - 0.7034)) \\ &= 1.8137 \end{aligned}$$

*with* $q = 0.7,\ C = 1.8257$

(e) A friend of you tells you that for solving the above pricing problem he used a binomial model with more intermediate time steps and calculated a slightly different value for the option price. Explain what happens if the number of time steps increases.

More time steps increase the accuracy of the solution. The binomial model converges to Black and Scholes model if the number of time steps goes to infinity.

4. **Bounds for Option Prices, Arbitrage**

The lower price bound of a European put (maturity date $T$ and strike price $K$) on a stock with price $S_t$ at time $t$ is given by:

$$P_t \geq \max\{Ke^{-r(T-t)} - S_t; 0\},$$

where $r$ denotes the risk-free interest rate and continuous compounding is used.

(a) Prove this inequality using the following two portfolios:

- Portfolio A: long European put with strike $K$
- Portfolio B: risk-free investment of $Ke^{-r(T-t)}$ and short position in the stock

Decompose $p_t \geq max\{Ke^{-r(T-t)} - S_t; 0\}$:

- $p_t \geq 0$ (holds always)
- $p_t \geq Ke^{-r(T-t)} - S_t$

Prove that $p_t \geq Ke^{-r(T-t)} - S_t$ using the two portfolios:

- Value of portfolio A (put long with strike $K$) at time $T$:

  (1) 0, if $S_T > K$

(2) $K - S_T > 0$, if $S_T < K$

- Value of portfolio B ($Ke^{-r(T-t)}$ long and $S_t$ short) at time $T$:

  (1) $K - S_T < 0$, if $S_T > K$

  (2) $K - S_T > 0$, if $S_T < K$

$\Rightarrow$ Portfolio A dominates Portfolio B

$\Rightarrow$ $p_t \geq max\{Ke^{-r(T-t)} - S_t; 0\}$

(b) Give a possible arbitrage strategy if the inequality is violated.

- Inequality is violated: $p_t < max\{Ke^{-r(T-t)} - S_t; 0\}$
- Arbitrage strategy
  - buy low (Put long)
  - sell high (Portfolio B short, i.e. buy stock, borrow money at the riskless rate)
- Payoff at $T$ if $S_T > K$: $S_T - K > 0$.
- Payoff at $T$ if $S_T < K$: $0$.

$\Rightarrow$ riskless profit with no (or even negative) initial investment!

5. **Hedging** **(18 Points)**

Mr. A. de Moivre works for a French bank and is very enthusiastic about the normal distribution. He believes that log returns are normally distributed and consequently uses the Black-Scholes model for option pricing. His supervisor asks him to construct a Delta-Gamma-Vega-Hedge in order to hedge a short position in a portfolio.

(a) What kind of instruments are suitable to set up a Delta-Gamma-Vega-Hedge? Explain your answer briefly. (2 points) Stock has Delta of one and Gamma/Vega of zero. Thus need at least two traded assets (options) that have exposure to gamma and vega.

(b) The portfolio is already delta neutral. The gamma is $-100$ and the vega is $-500$. Mr. A. de Moivre identifies two options to construct the hedge.

| | Delta | Gamma | Vega |
|---|---|---|---|
| Option 1 | 0.5 | 0.6 | 1.8 |
| Option 2 | -0.6 | 0.8 | 1.5 |

i. Give the exact position in each instrument to make the portfolio gamma and vega neutral. (5 points)

$$-100 + 0.6w_1 + 0.8w_2 = 0$$
$$-500 + 1.8w_1 + 1.5w_2 = 0$$

Which yields: $w_1 = 462.9630$, and $w_2 = -222.2222$, i.e. long option 1 and short option 2. $w_1 = 462.9630$, and $w_2 = -222.2222$.

ii. What is the delta of the new portfolio (after the addition of the positions in the two options)? How can you make the portfolio delta neutral without changing the other sensitivities? (2 points)

The delta of the portfolio is now 364.8148. Use a position in the underlying (short). The delta of the portfolio is now 364.8148.

(c) A friend approaches Mr. A. de Moivre and tells him that he does not understand Vega-Hedging.

i. Explain why the Vega of a long position in a plain vanilla option is always positive. (4 points)

Vega measures the sensitivity of the option price with respect to the volatility of the underlying. Since the option price equals the discounted expected risk-neutral payoff and the payoff is a convex function of the stock price at maturity, the put price increases if the volatility of the underlying increases (we take the expectation of a convex payoff).

ii. Comment on the statement:
"I think that Vega-Hedging makes no sense in the Black-Scholes model!"
Carefully explain why Mr. A. de Moivre should or should not engage in Vega-Hedging. (5 points)

Statement is true: $\sigma$ is a constant in the BS-Model. However, empirically volatility changes over time and a vega hedge performs better than a simple delta hedge. Could proxy for more sophisticated option pricing models.

6. **Applying Ito's Lemma**

The stock price $S_t$ follows a geometric Brownian motion (GBM), i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with constant parameters $\mu$ and $\sigma$.

Derive the dynamics of the following expressions and simplify the expressions as much as possible:

Ito's Lemma:
Suppose that $S_t$ follows the stochastic process $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$. Then for any twice differentiable function $F$, the process $F(S_t, t)$ follows

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}(dS_t)^2$$

where $(dt)^2 = 0$, $dt \cdot dW = 0$ and $(dW^2) = dt$

(a) $d(S_t^{-2})$ with $F = S_t^{-2}$

$$\frac{\partial F}{\partial S} = -2S_t^{-3}, \qquad \frac{\partial^2 F}{\partial S^2} = 6S^{-4}, \qquad \frac{\partial F}{\partial t} = 0$$

With $F = S^{-2}$ we obtain

$$\begin{aligned} dF_t &= -2S_t^{-3} dS_t + \frac{1}{2} 6S^{-4}\sigma^2 S_t^2\, dt \\ &= -2S_t^{-3}(\mu S_t dt + \sigma S_t dW_t) + 3\sigma^2 S_t^{-2}\, dt \\ &= -2S_t^{-2}(\mu dt + \sigma dW_t) + 3\sigma^2 S_t^{-2}\, dt \\ &= (-2\mu + 3\sigma^2)F_t dt + -2\sigma F_t dW_t) \end{aligned}$$

(b) $d(\ln(S_t^2))$ with $F = \ln(S_t^2)$

$$\frac{\partial F}{\partial S} = 2S_t \frac{1}{S_t^2} = \frac{2}{S_t}, \qquad \frac{\partial^2 F}{\partial S^2} = -2S^{-2}, \qquad \frac{\partial F}{\partial t} = 0$$

With $F = \ln(S_t^2)$ we obtain

$$\begin{aligned} dF_t &= \frac{2}{S_t} dS_t + \frac{1}{2} - 2S^{-2}\sigma^2 S_t^2\, dt \\ &= \frac{2}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \sigma^2 dt \\ &= 2(\mu dt + \sigma dW_t) - \sigma^2 dt \\ &= (2\mu - \sigma^2)dt + 2\sigma dW_t) \end{aligned}$$

Now assume that $X_t$ is governed by the following stochastic process

$$dX_t = a(b - X_t)dt + c\, dW_t,$$

where $a$, $b$, and $c$ are positive constants, and $W_t$ is a Wiener Process.

(c) Is the above introduced process useful to describe the behavior of stock prices?

No. This is a mean reverting process and stock prices are not mean reverting. Process can take negative values ...

(d) Use Ito's Lemma to determine the sde of the stochastic process of $X_t e^{at}$.

Calculate the partial derivatives and apply Ito's Lemma with $f(x,t) = xe^{at}$

$$\begin{aligned} dX_t e^{at} &= aX_t e^{at} dt + e^{at} dX_t \\ &= e^{at} abdt + ce^{at} dW_t \end{aligned}$$

7. **Black-Scholes Pricing versus the Physical World**

Consider the following stock price processes

$$dS_t^i = \mu^i S_t^i \, dt + \sigma^i S_t^i \, dW_t^i$$

where $i = 1, 2$ and assume that $dW_t^1$ and $dW_t^2$ are uncorrelated.

(a) Compute the distribution of $\ln S_t^i$ for each $i$.

Use Ito to compute $d \ln S_t^i$ and solve the SDE. The result is $\ln S_t^i \sim N(\ln S_0 + (\mu^i - 0.5(\sigma^i)^2)t; (\sigma^i)^2 t)$

(b) Given the parameters $\mu^1 = 0.075$, $\mu^2 = 0.015$, $\sigma^1 = \sigma^2 = 0.1$ and $S_0^1 = S_0^2 = 100$ compute the probability that $S_T^i > 100$ for $T = 1$, for $i = 1, 2$.

We have $P(S_T^i > 100) = P(\ln S_T^i > \ln 100)$, after standardizing the normally distributed variables we have to search for $P(z^1 < 0.7) = 0.7580$ and $P(z^2 < 0.1) = 0.5398$ .

(c) Consider two identical calls on each of the two stocks with $K = 100$ and $T = 1$. (Assume a complete market.) Which call would be more expensive? Explain your answer. Use the result from 7b in your explanation.

The calls will have the same price. In the question above we computed the probabilities for the call ending up in the money under the P-measure. Pricing an option can be conducted by means of replication. For that we only assume that the market is arbitrage-free and investors are greedy. In particular we do not have to say anything about the risk preferences of investors on the market. For setting up a replication portfolio to price an option we do not need the expected return of the underlying under the physical measure and, therefore, $\mu$ has no implications for pricing the option. Only the volatility of the underlying will influence the pricing,and since the volatilities of both underlyings are equal also the prices are.

8. **Monte Carlo Simulation**

(a) After 5000 simulation runs you obtain an Monte Carlo estimator for the option price $\widehat{C_0} = 20.05$ and a standard deviation of $\widehat{\sigma} = 10.02$. What can you say about the price estimator?

We can asses precision of the price estimator by calculating the confidence interval. First, calculate standard error: $\sigma_{se} = \frac{\widehat{\sigma}}{\sqrt{N}} = 0.1417$. Confidence interval is then given as $20.05 \pm 1.96 * 0.1417 = [19.7723; 20.3277]$. Quite large confidence interval. One would like to have precision with 95% confidence at least on cent level.

(b) You want to reduce the variance of the estimator by means of the control variate technique. The simulation results in a value $\widehat{D_0} = 25.25$ for the control derivative that is positively correlated with the option to be priced $C$. The analytical value if the control derivative is $D = 25$. Given your information perform an adequate ad-hoc adjustment, assuming that $\alpha^* = 1$, to the the price estimator $\widehat{C_0}$.

Adjustment of price estimator by means of control variate technique is given by: $\widehat{C_0} + \alpha^*(\widehat{D_0} - D_0)$. From estimation results for $D$ we learn that there is an positive estimation error; in other words the estimator is too large. In general that means, that the estimator for $C$ should be adjusted downwards. However, we do not know optimal adjustment factor $\alpha^*$, therefore, we assume $\alpha = 1$ and adjust price estimator by an estimation error of the same size as for the control derivative $D$. That is, adjusted estimation error $\widehat{C_0^{ad}} = 20.05 - (25.25 - 25) = 19.8$.

(c) Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.

Pricing American-style options is typically performed by a backwards in time recursive procedure to consider optimal exercise decisions. Standard Monte Carlo simulation performs forward simulation and therefore, exercise decision cannot be considered correctly.

9. **Credit Risk and Credit Derivatives**

(a) Suppose that:

i. The yield on a 5-year risk-free bond is 7% p.a. .
ii. The yield on a 5-year corporate bond issued by company X is 9.5% p.a. .
iii. A 5-year credit default swap providing insurance against company X defaulting costs 150 basis points per year.

What arbitrage opportunity is there in this situation?

Idea: combination of CDS and corporate bond results in risk-free bond

Set up arbitrage portfolio:

First part of arbitrage portfolio consists of long CDS (shorting default risk of writer of corporate bond) plus holding corporate bond. This part of the arbitrage portfolio earns $-1.5\% + 9.5\% = 8\%$ of the notional of the underlying bond contract p.a. .

Second part consists of shorting risk-free bond (financing position in corporate bond). This part of the arbitrage portfolio "earns" $-7\%$ of notional of the underlying bond contract p.a. .

In sum arbitrage portfolio earns $1\%$ risk-free arbitrage return p.a. .

(b) Explain how you would expect the returns offered on the various tranches in a CDO to change when the correlation between the bonds in the portfolio increases.

Assume each reference entity has a probability of default of x% (example: 2%) over a specific period of time (5 years). Let us look at extreme cases. If correlation is 1 then essentially all reference entities are the same and the probability of observing 1 or more defaults is 2%. If correlation is zero the probability that at least one of all the reference entities defaults is quite large (much larger than x%) and the probability of observing a large number of defaults is very small (much smaller than x%). That means for CDOs that if correlation increases the junior tranches become less risky (returns go down) and the senior tranches become more risky (returns go up). In the limit where the default correlation is perfect and the recovery rate is zero, the tranches are all equally risky and all tranches pay the same interest rates.

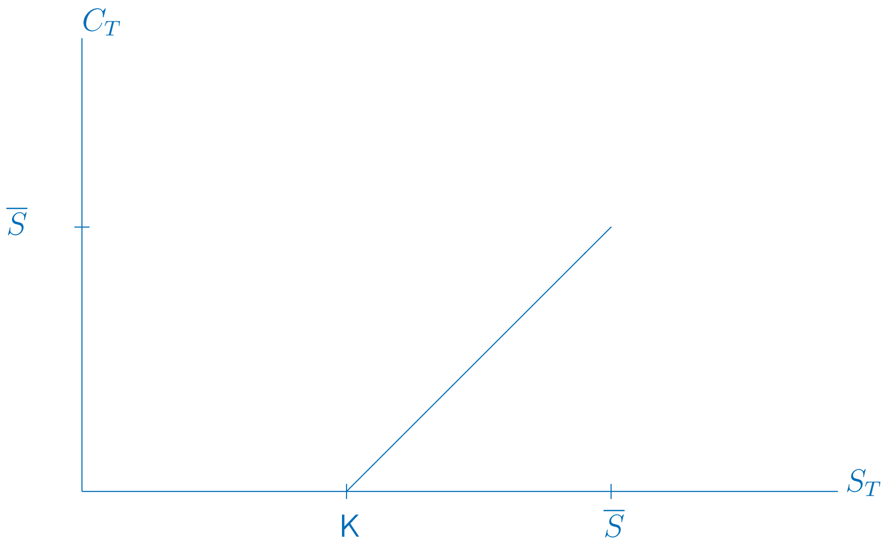
10. **Exotic Pricing in Continuous Time**

Assume the Black-Scholes model holds. Consider the following derivative security with maturity $T$ and payment structure:

$$C_T = (S_T - K) * I_{\{\overline{S} > S_T > K\}}$$

with $K < \overline{S}$ and $I$ denotes the indicator function

(a) Draw the payoff of the security in a diagram. Do not forget to label axes.




(b) How would you value this contract? Give a pricing equation.

$$\begin{aligned}
\frac{C_0}{N_0} &= E_0^N\left[\frac{C_T}{N_T}\right] \\
&= E_0^N\left[\frac{(S_T-K)\cdot I_{\{\overline{S}>S_T>K\}}}{N_T}\right] \\
&= E_0^N\left[\frac{S_T\cdot I_{\{\overline{S}>S_T>K\}}}{N_T}\right] - E_0^N\left[\frac{K\cdot I_{\{\overline{S}>S_T>K\}}}{N_T}\right] \\
&= \widetilde{E}_0\left[I_{\{\overline{S}>S_T>K\}}\right] - e^{-rT}K\widehat{E}_0\left[I_{\{\overline{S}>S_T>K\}}\right] \\
&= \widetilde{P}_0\left[\overline{S}>S_T>K\right] - e^{-rT}K\widehat{P}_0\left[\overline{S}>S_T>K\right] \\
&= \{\widetilde{P}_0\left[S_T<\overline{S}\right] - \widetilde{P}_0\left[S_T<K\right]\} - e^{-rT}K\cdot\{\widehat{P}_0\left[S_T<\overline{S}\right] - \widehat{P}_0\left[S_T<K\right]\} \\
\Rightarrow C_0 &= S_0\cdot\{\widetilde{P}_0\left[S_T<\overline{S}\right] - \widetilde{P}_0\left[S_T<K\right]\} - e^{-rT}K\cdot\{\widehat{P}_0\left[S_T<\overline{S}\right] - \widehat{P}_0\left[S_T<K\right]\}
\end{aligned}$$

(c) A 'pay later' option is a contract with the payoff function $C_T^{PL} = (S_T - K - \pi)I_{\{S_T\geq K\}}$. Here $\pi$ denotes the premium, which is only paid if the option is ultimately exercised. Compute $\pi$ such that the contract has zero value initially.

$C_0^{PL} \overset{!}{=} 0$; $\pi = ?$

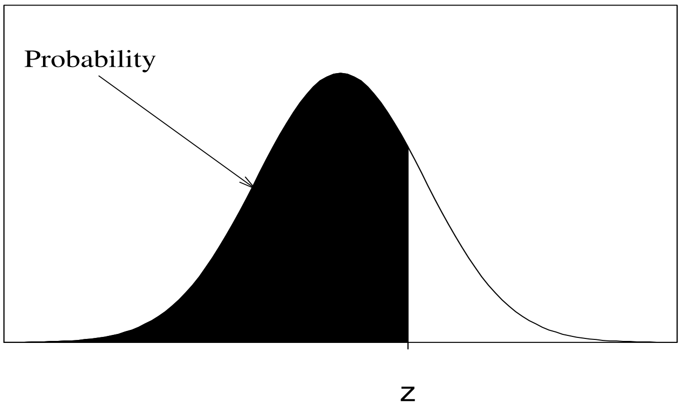
$$\begin{aligned}
\frac{C_0^{PL}}{N_0} &= E_0^N\left[\frac{C_T}{N_T}\right] \\
&= E_0^N\left[\frac{(S_T - K - \pi)I_{\{S_T \geq K\}}}{N_T}\right] \\
&= E_0^N\left[\frac{(S_T - K)I_{\{S_T \geq K\}}}{N_T}\right] - \pi \cdot \widehat{E}_0\left[\frac{I_{\{S_T \geq K\}}}{e^{rT}}\right] \\
\Rightarrow C^{PL} &= S_0 N(d_1) - e^{-rT} K N(d_2) - e^{-rT}\pi N(d_2)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
0 &= S_0 N(d_1) - e^{-rT} K N(d_2) - e^{-rT}\pi N(d_2) \\
\Leftrightarrow \pi &= \frac{S_0 N(d_1) - e^{-rT} K N(d_2)}{e^{-rT} N(d_2)}
\end{aligned}$$

## Probabilities for the standard normal distribution

Table entry for $z$ is the probability lying to the left of $z$




| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |