Complex Analysis   Final Exam 2022   Vrije Universiteit Amsterdam

- 30 May 2022, 12:15–14:15.
- Use of calculators, books or notes is not allowed. Motivate your answers.
- Each (sub-)problem is worth 1 point. Total points = 11. Grade = $1 + 9 \cdot \frac{\text{points}}{11}$.

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1. Determine the Laurent expansion, around $z_0 = 0$, of the following functions:
   (a) $z/(z+2)$, for $|z| < 2$,
   (b) $z/(z+2)$, for $|z| > 2$,
   (c) $z/(z+2)^2$, for $|z| > 2$.

2. Show that
$$e^z = e + e\sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^n$$
for all complex numbers z.

3. Let $f(z) = (\cosh(2z) - 1 - 2z^2)/z^4$ for $z \neq 0$. Determine the constant a such that
$$g(z) = \begin{cases} f(z) & \text{if } z \neq 0 \\ a & \text{if } z = 0 \end{cases}$$
is analytic at $z_0 = 0$.

4. Compute the following integrals.
   (a)
$$\int_C \tan(z)dz,$$
   where C is the circle of radius 2 centered at the origin, oriented counter-clockwise.
   (b)
$$\int_C \sin(1/z)z^4 dz,$$
   where C is the unit circle centered at the origin, oriented counter-clockwise.
   (c)
$$\int_C \frac{z^5\cos(1/z)}{(z^3+1)^2}dz,$$
   where C is the circle of radius 2 centered at the origin, oriented counter-clockwise.
   (d)
$$\int_0^{\infty} \frac{\cos(x)}{x^2+b^2}dx,$$
   where $b > 0$ is a real constant.

5. (a) Assume that $f(z)$ has a pole of order $m = 1$ at z_0. Prove that $f(z) = \phi(z)/(z - z_0)$ where $\phi(z)$ is analytic and nonzero at z_0.
   (b) Let $f(z)$ be an entire function such that $g(z) = f(1/z)$ has a removable singularity at $z_0 = 0$. Show that $f(z)$ must be a constant function.