# Complex Analysis: Final Exam

**Vrije Universiteit Amsterdam**

- 21 May 2021, 18:45 - 20:45 (+ 10 minutes submission time).
- Use of calculators, books or notes is not allowed. Motivate your answers.
- Each problem is worth 1 point. Total points = 11.

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1. Compute the Laurent expansion (around $z_0 = 0$) of the following functions.
   (a) $f(z) = \frac{1+2i}{(z+2i)(z-1)}$ on the domain $\{z \in \mathbb{C} : |z| < 1\}$.
   (b) $f(z) = \frac{1+2i}{(z+2i)(z-1)}$ on the domain $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
   (c) $f(z) = \frac{1+2i}{(z+2i)(z-1)}$ on the domain $\{z \in \mathbb{C} : 2 < |z| < \infty\}$.

2. (a) Assume that $f$ has an isolated singularity at $z = z_0$. Use the Laurent expansion of $f$ around $z_0$ to define the notion of
      i. a removable singularity of $f$ at $z_0$.
      ii. a pole of order $m$ of $f$ at $z_0$,
      iii. an essential singularity of $f$ at $z_0$.
   (b) Let
   $$f(z) = \begin{cases} (\sinh(z) - z)/z^3 & \text{for } z \neq 0 \\ \frac{1}{6} & \text{for } z = 0. \end{cases}.$$
   Show that $f(z)$ is an entire function.

3. Compute the following integrals.
   (a)
   $$\int_C z^2 e^{1/z} dz,$$
   where $C$ is the unit circle, oriented counter-clockwise.
   (b)
   $$\int_C \frac{\cos(z)}{z(z-2)} dz,$$
   where $C$ is the unit circle, oriented counter-clockwise.
   (c)
   $$\int_C \frac{z^6}{(z^2 - i)(z-2)^5} dz,$$
   where $C$ is the circle of radius 3 centered at the origin, oriented counter-clockwise.
   (d)
   $$\int_0^\infty \frac{\cos(x)}{(x^2+1)^2} dx.$$
   (e)
   $$\int_0^\infty \frac{x}{x^3+1} dx.$$

4. Assume that $g(z) = f(z^2)$ where $f$ is analytic at $z = 0$. Show that
   $$0 = g'(0) = g'''(0) = \ldots = g^{(2n-1)}(0) = \ldots.$$