Basics of Graph Theory

Exercise 1.7. How many 2-regular graphs exist with $V = \{1, 2, 3, 4, 5\}$?

Solution: One graph: A cycle on all the 5 points (usually denoted by C_5).

Exercise 1.8. How many 3-regular graphs exist with $V = \{1, 2, 3, 4, 5\}$?

Solution: None. For a proof see Exercise 1.16.

Exercise 1.11. How many edges has a 5-regular graph on 16 vertices?

Solution: $5 \cdot 16/2 = 40$.

Exercise 1.12. How many edges has a k-regular graph on n vertices?

Solution: $kn/2$.

Exercise 1.16. Prove that every graph has an even number of points with odd degree.

Solution: When we take the sum over all vertex degrees then each edge is counted twice:

$$\sum_{v \in V} d(v) = 2|E|$$

Hence, the sum of the degrees is even. Hence, there must be an even number of odd degree vertices.

Exercise 1.19. Prove that a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ has a vertex with degree $\leq 2m/n$ and a vertex with degree $\geq 2m/n$.

Solution: From Exercise 1.16 it follows that the average degree is exactly $2m/n$. Hence, there must be a vertex with degree $\leq 2m/n$ and a vertex with degree $\geq 2m/n$.

Exercise 1.21. Prove that every graph $G = (V, E)$ with $|V| \geq 2$ has two vertices of the same degree.

Solution: For any vertex v we have $d(v) \in \{0, 1, \ldots, n-1\}$. Further, there cannot be two vertices u, v with $d(u) = 0$ *and* $d(v) = n - 1$. So, there are at most $n - 1$ different degrees. Since we have more vertices than different degrees, there must be at least two vertices that have the same degree.

Exercise 1.26. For which values of m and n is $K_{m,n}$ regular?

Solution: Only regular if $m = n$.

Exercise 1.41. How many paths are there from vertex 1 to vertex 3 in K_3?

Solution: Two paths: $(1, 3)$ and $(1, 2, 3)$.

Exercise 1.42. How many paths are there from vertex 1 to vertex n in K_n?

Solution:

Number of paths of length 1:	1
Number of paths of length 2:	$n - 2$
Number of paths of length 3:	$(n - 2)(n - 3)$
$\vdots$	$\vdots$
Number of paths of length i:	$(n - 2)(n - 3) \cdots (n - i)$
$\vdots$	$\vdots$
Number of paths of length n-1:	$(n - 2)(n - 3) \cdots 1$

Exercise 1.44. Prove that that a graph of which each vertex has degree at least k, has a path of length k.

Solution: Start in any vertex, say v_0, and construct a path as follows. If the path so far is $v_0, v_1, \ldots, v_i$ with $i \leq k - 1$ then let v_{i+1} be a neighbor of v_i that is not in $\{v_0, v_1, \ldots, v_{i-1}\}$. This is possible since $d(v_i) \geq k \geq i + 1$. Hence, as long as the length of the path is less than k, we can extend it. This results in a path of length at least k.

Exercise 1.47. Prove that every connected graph on n vertices contains at least $n - 1$ edges.

Solution: We prove this by induction on n. The statement is true for $n = 1$. Now assume it holds for all $n' < n$. If all degrees are at least 2, then the number of edges is at least $2n/2 = n$ (Count as in ex. 1.16). If not all degrees are at least 2 then there is a vertex v of degree 1 (zero is not possible since the graph is connected). Remove v and its adjacent edge form the graph. The remaining graph is connected and has $n - 1 < n$ vertices. So by induction, it has at least $(n - 1) - 1 = n - 2$ edges. Hence, the original graph has at least $n - 1$ edges.

Exercise 1.48. Does there exist a non-connected graph on 6 vertices containing 11 edges?

Solution: No. For a proof, see the next exercise.

Exercise 1.50. Prove that every non-connected graph on n vertices contains at most $\frac{1}{2}(n-1)(n-2)$ edges.

Solution: Let G be non-connected. Then we can partition the vertex set V in two sets, say S and $V \setminus S$, such that there are no edges between S and $V \setminus S$. Let $|S| = k$. Then, $|V \setminus S| = n - k$. The number of missing edges is at least $k(n-k)$. The number is minimal for $k = 1$ and $k = n - 1$. Hence, the number of missing edges is at least $n - 1$. The number of edges is at most $n(n-1)/2 - (n-1) = (n-1)(n-2)/2$.

Exercise 1.59. Prove (from the definition) that if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two distinct components of G then $V_1 \cap V_2 = \emptyset$.

Solution: Assume $V_1 \cap V_2 \neq \emptyset$. Then the graph $G'' = (V_1 \cup V_2, E_1 \cup E_2)$ is connected and both G_1 and G_2 are subgraphs of G''. Since G_1 and G_2 are different, at least one of these two graphs is not the same as G''. But then that graph is not a component by the given definition.

Exercise 1.63. Prove that a graph $G = (V, E)$ with each vertex having degree at least $\frac{1}{2}(n-1)$ is connected.

Solution: Assume it is not connected. Then it has a component with at most $n/2$ vertices. Each vertex in that component has degree at most $n/2 - 1 < (n-1)/2$. A contradiction. Hence, it must be connected.

Exercise 1.64. Prove that a graph $G = (V, E)$ has at least $|V| - |E|$ components.

Solution: Let k be the number of components and let n_i be the number of vertices in component i, $(i = 1, \ldots, k)$. Component i has at least $n_i - 1$ edges (see exercise 1.47). Hence,

$$|E| \geq \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i \; - k = |V| - k \quad \Rightarrow \quad k \geq |V| - |E|.$$

Exercise 1.65. Prove that a graph with exactly two vertices with odd degree must contain a path between these two vertices.

Solution: Let u and v be the vertices of odd degree. Assume there is no path between u and v. Then u and v are in different components. Each of those components has exactly one vertex of odd degree. This is not possible (by exercise 1.16).

Exercise 1.67. Let G be a graph for which every vertex has a degree of at least 2. Prove that G contains a circuit.

Solution: Make a walk $v_1, v_2, \ldots$ in the graph such that for any $i \geq 2$, $v_{i+1} \neq v_{i-1}$. Since the graph has only a finite number of points we must have that $v_i = v_j$ for some pair $i < j$. For the first moment that this happens, the cycle $v_i, v_{i+1}, \ldots, v_j$ is a circuit.
(*Note that the restriction $v_{i+1} \neq v_{i-1}$ is necessary since, for example, a walk in a tree does not lead to a circuit.*)

Exercise 1.75. Prove that between any pair of vertices in a tree there is exactly one path.

Solution: If there are two paths then there must be a circuit. However, a tree has no circuits.

Exercise 1.76. Prove that every tree with at least two vertices contains a leaf (cf. Exercise 1.67).

Solution: If it has no leaf then every vertex has degree at least 2. By Exercise 1.67, it has a circuit. However, a tree has no circuit.

Exercise 1.77. Derive from the previous exercise that every tree on n vertices has exactly $n-1$ edges.

Solution: We prove it by induction on n. It is true for $n = 1$. Now consider a tree T on $n \geq 2$ leaves and assume that the statement holds for all $n' \leq n$. By the previous exercise T must have a leaf v. Deleting v and its adjacent edge from the tree gives a tree on $n-1$ vertices. By induction, it has exactly $n-2$ edges. Hence, T has $n-1$ edges.

Exercise 1.79. Prove that a forest on n vertices consisting of k components contains exactly $n-k$ edges.

Solution: Let n_i be the number of vertices in component i ($i = 1, \ldots, k$). By Exercise 1.78, component i has $n_i - 1$ edges.

$$|E| = \sum_{i=1}^{k} (n_i - 1) = n - k.$$

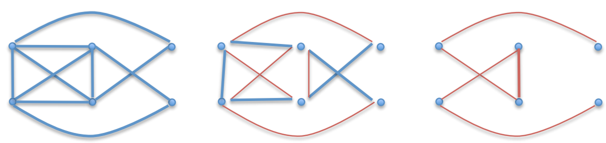
Exercise 1.80. Prove that every tree with at least two vertices contains at least two leaves.

Solution: Let $v_1, v_2, \ldots, v_k$ be a longest path in the tree. If $d(v_1) \geq 2$ then

v_1 has a neighbor that is not on the path. But then we can extend the path. So the degree of v_1 is one. Similarly, we must have $d(v_k) = 1$.

Exercise 1.86 Let G be an Euler graph with an even number of edges. Let $d_1, d_2, \ldots, d_n$ be the degrees of the points. Show that there exists a subgraph with degrees $d_1/2, d_2/2, \ldots, d_n/2$.

Solution: An Euler graph has an Euler tour. Color the edges with alternate colors. Remove all edges of one of the two colors. The remaining subgraph has the given property.

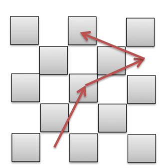


Exercise 1.88. Let G be a connected graph with exactly two points of odd degree. Use Euler's Theorem to prove that G contains a walk that traverses each edge exactly once.

Solution: Let u and v have odd degree. Adding the edge $\{u, v\}$ makes the graph Eulerian. By Euler's theorem it has an Euler tour. Deleting $\{u, v\}$ from the tour gives a path that traverses each edge of graph G exactly once.

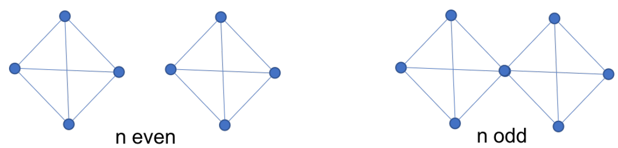
Exercise 1.90 Let n be an odd number. Show that on an $n \times n$ chess board, it is not possible for a knight (horse) to move over the board, hitting each square exactly once, while starting and ending in the same square.

Solution: A knight moves from white to black and vice versa. Since n is odd, the number of squares, n^2, is odd. If it starts on black, then it is on white after n^2 moves. But then it cannot be back at its starting point.



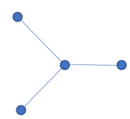
Exercise 1.91. Show that for each n there exists a graph on n vertices such that each vertex has degree at least $\frac{1}{2}n - 1$ and such that it is not a Hamilton graph.

Solution: If n is even then take two complete components on $n/2$ vertices. If n is odd then take two complete graphs on $(n-1)/2$ vertices and add a vertex v and connect it to all other vertices.



Exercise X.1 Give an example of a connected graph with an even number of vertices that does not have a perfect matching.

Solution:



Flow exercises. Solutions.

Solution 1:
This is a special case of Theorem 1. By definition value(f) is the total (nett) flow leaving s:

$$\text{value}(f) = \sum_{v \in V} (f_{sv} - f_{vs}).$$

On the other hand, applying Theorem 1 with $U = V \setminus \{t\}$ gives.

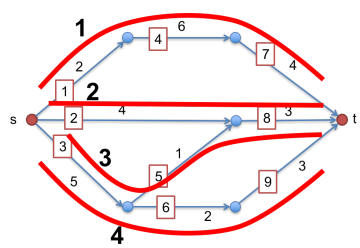
$$\text{value}(f) = \sum_{u \in U} \sum_{v \in V \setminus U} (f_{uv} - f_{vu}) = \sum_{u \in V \setminus \{t\}} (f_{ut} - f_{tu}) = \sum_{u \in V} (f_{ut} - f_{tu}).$$

(The last equality holds since $f_{tt} = 0$.)

Solution 2:
Make the following network. Take vertices $v_1, v_2, \ldots, v_p$ where v_i corresponds with family i. For each table j, take a vertex w_j. Further add points s and t. There is an arc (s, v_i) with capacity a_i for each $i \in \{1, \ldots, p\}$. There is an arc (w_j, t) with capacity b_j for each $j \in \{1, \ldots, q\}$. For each pair i, j, there is an arc (v_i, w_j) with capacity 1.

An upper bound on the maximum flow vale is $\sum_i a_i$ since that is the maximum flow that can leave s. If there exists a flow of value $\sum_i a_i$ then this immediately give a solution to the dinner problem since, by Theorem 3, the flow on each arc (v_i, w_j) is either 0 or 1. If the flow value on (v_i, w_j) is one then a person from family i is seated at table j.

Solution 3:
(a) The network has four s-t paths that we label as shown.



$$
\begin{array}{lrcrcrcrll}
\max & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
s.t. & x_1 & & & & & & & \leq 2 & \text{(edge 1)} \\
 & & & x_2 & & & & & \leq 4 & \text{(edge 2)} \\
 & & & & & x_3 & + & x_4 & \leq 5 & \text{(edge 3)} \\
 & x_1 & & & & & & & \leq 6 & \text{(edge 4)} \\
 & & & & & x_3 & & & \leq 1 & \text{(edge 5)} \\
 & & & & & & & x_4 & \leq 2 & \text{(edge 6)} \\
 & x_1 & & & & & & & \leq 4 & \text{(edge 7)} \\
 & & & x_2 & + & x_3 & & & \leq 3 & \text{(edge 8)} \\
 & & & & & & & x_4 & \leq 3 & \text{(edge 9)}
\end{array}
$$

(b) The maximum flow value is 7. A max flow is $f_1 = 2, f_2 = 3, f_3 = 2, f_4 = 2, f_5 = 0, f_6 = 2, f_7 = 2, f_8 = 3, f_9 = 2$.

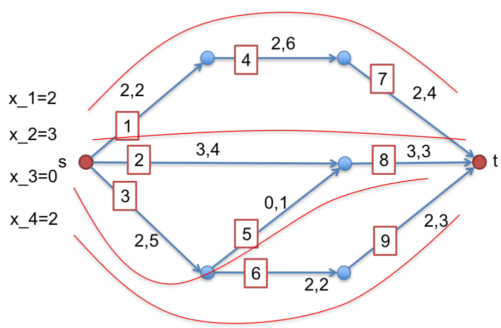


Figure 1: The first value of each pair of numbers is the flow and the second the capacity of the arc. The value x_i is the flow on path i.

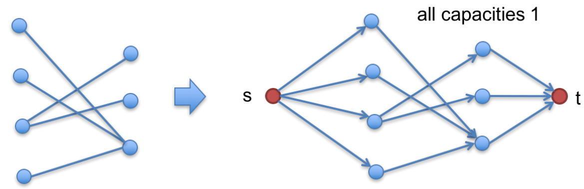
(c)

$$
\begin{array}{lrcrcrcrcrcrcrcrcrcrll}
\min & 2y_1 & + & 4y_2 & + & 5y_3 & + & 6y_4 & + & y_5 & + & 2y_6 & + & 4y_7 & + & 3y_8 & + & 3y_9 & & \\
s.t. & y_1 & & & & & + & y_4 & & & & & + & y_7 & & & & & \geq 1 & \text{(path 1)} \\
 & & & y_2 & & & & & & & & & & & + & y_8 & & & \geq 1 & \text{(path 2)} \\
 & & & & & y_3 & & & + & y_5 & & & & & + & y_8 & & & \geq 1 & \text{(path 3)} \\
 & & & & & y_3 & & & & & + & y_6 & & & & & + & y_9 & \geq 1 & \text{(path 4)}
\end{array}
$$

(d) $y_1 = 1, y_6 = 1, y_8 = 1$, and $y_2 = y_3 = y_4 = y_5 = y_7 = y_9 = 0$.

(e) The optimal dual solution corresponds with a minimum cut: The edges in the cut have value $y_i = 1$.

Solution 4:



Solution 5:

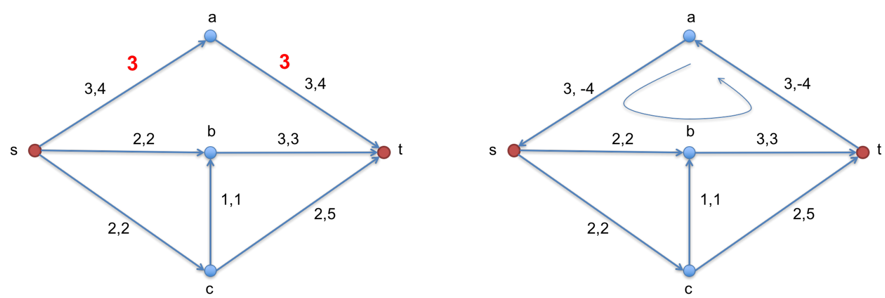


Figure 2: Left: Flow f_1 of cost 24. Right: Residual(f_1). It has a negative cost cycle, (s, b, t, a, s), of cost $2 + 3 - 4 - 4 = -3$ and the minimum capacity on the cycle in the residual is 2.

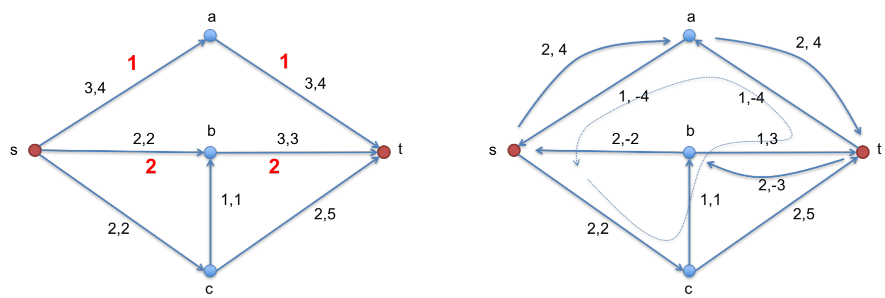


Figure 3: Left: Flow f_2 of cost 18. Right: Residual(f_2). It has a negative cost cycle, (s, c, b, t, a, s), of cost $2 + 1 + 3 - 4 - 4 = -2$ and the minimum capacity on the cycle in the residual is 1.

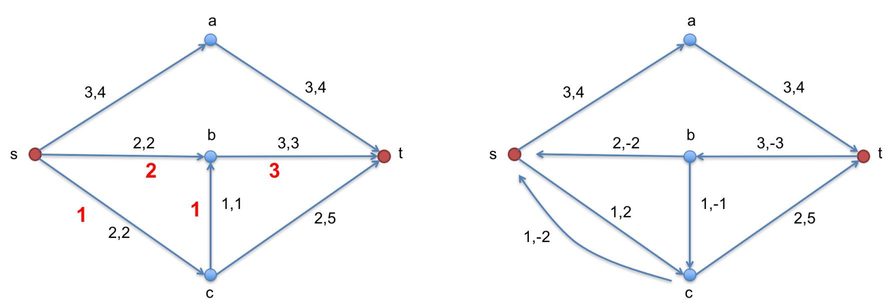
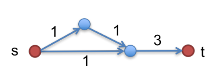


Figure 4: Left: Flow f_3 of cost 16. Right: Residual(f_3). It has no negative cost cycle. Hence f_3 is a minimum cost flow.

Solution 6:
(a) True. Divide all capacities by 2. Theorem 3 says that there is an optimal flow with integer flow values f_a on each arc. Now multiply all f_a by two.
(b) Not true. This is a counter example.



Solution 7:
(a) No. Take for example a path (s, v, t) with capacity 1 on each of the two arcs.
(b) Yes. If we decrease the capacity of any arc a in the minimum cut then the capacity of the minimum cut decreases. Since MaxFlow=MinCut, the maximum flow value decreases as well.