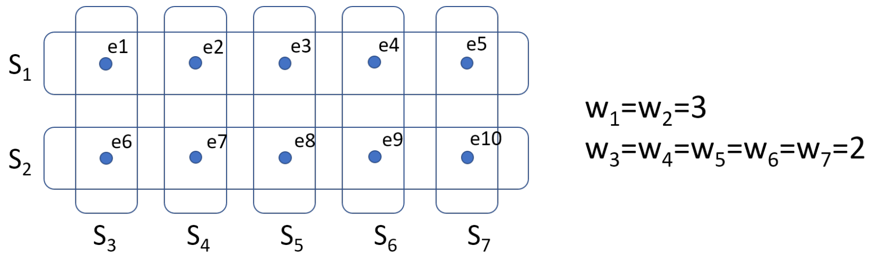
Exam Combinatorial Optimization, 27 October 2022.

- It is **not** allowed to use any books, notes, or calculator. Just pen and paper.
- You may also obtain points for partial answers.
- Clearly indicate where your final answer is in case you make some computations.

1a	1b	1c	2	3a	3b	4a	4b	5a	5b	6a	6b	Σ
2	2	3	8	2	6	2	5	2	5	3	5	45

Question 1 (7 points)
The figure below is an instance of the weighted set cover problem. In fact, it is an instance of the weighted vertex cover problem since each element appears in exactly two sets.



(a) (2pt) Draw the graph that corresponds to this vertex cover instance.

Chapter 1 gives a primal-dual algorithm for the weighted set cover problem. Remember that this algorithm constructs the primal and dual solution step by step without actually solving the dual to optimality.

(b) (2pt) Formulate this dual for this instance. (Use dual variables y_i.)

(c) (3pt) Now apply the primal-dual algorithm to this instance. Go through the variables in the order $y_1, y_2, \ldots$. Show how the values of the dual variables change in each iteration. Give the final dual solution and the set cover. Also, give the value of the set cover found.

Question 2 (8 points)
Consider the problem of minimizing makespan on m identical parallel machines: $Pm||C_{\max}$. Let n be the number of jobs and let p_j be the length of job j for $j = 1, 2 \ldots, n$.

Show that the list scheduling algorithms is a $2 - 1/m$ approximation algorithm for this problem.

(Describe clearly what lower bounds on the optimal value are used. A picture may be useful here. Some credits are given for proving only 2 instead of $2 - 1/m$.)

Question 3 (8 points)
This question is about Christofides' algorithm for the metric Traveling Salesman Problem.

(a) (2pt.) Describe Christofides' algorithm for the metric TSP briefly by filling in the blanks of step (1) to step (4).

(1) Find a minimum ...

(2) Find a ...

(3) Add the ... to the ...

(4) Find ...

(5) Cut short.

(b) (6pt.) Show that the approximation ratio of Christofides' algorithm algorithm is no more than $3/2$. Do this is in three steps:
First, argue that the cost of the subgraph computed in step (1) is no more than OPT (the optimal TSP value).
Then argue that the cost of the subgraph computed in step (2) is not more than OPT$/2$.
Finally, argue that the cost does not increase in step (5).

(N.B. You do not need to argue about step (4) and nor about the running time. Drawings are useful here.)

Question 4 (7 points)
The *conjunctive* maximum satisfiability problem is a variant of the maximum satisfiability problem that we have seen in this course. Here the clauses are a conjunction of literals. That means, a clause is *satisfied* if and only if <u>all</u> its literals are true. Here is an example:

$$C_1 = x_1 \wedge x_2, \qquad C_2 = x_1 \wedge \overline{x_3}, \qquad C_3 = \overline{x_1} \wedge x_3, \qquad C_4 = \overline{x_2} \wedge \overline{x_3}, \qquad C_5 = x_2 \wedge \overline{x_3}.$$

For example, clause C_2 is satisfied (is true) if and only if $x_1 =$ True and $x_3 =$ False.

(a) (2pt.) What is the expected number of satisfied clauses in the example above when you assign each variable independently the value 'True' with probability 0.5? Show your computation.

(b) (5pt.) Now suppose that you apply the method of conditional expectations to derandomize this approach. Assume that you go through the variables in the order x_1, x_2, x_3 in the example above. Then what assignment do you get and how many clause are satisfied? Show your computation.

Question 5 (7 points)
Below is the ILP for the Uncapacitated Facility Location (UFL) problem.

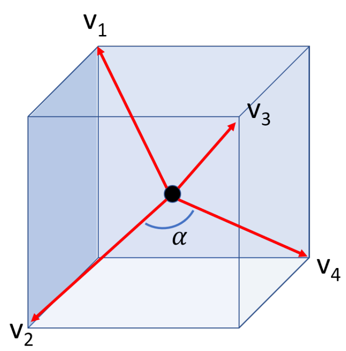
$$
\begin{array}{llll}
\text{(ILP)} & \min & Z = \sum_{i\in F} f_i y_i + \sum_{i\in F, j\in D} c_{ij} x_{ij} & \\
& s.t. & \sum_{i\in F} x_{ij} = 1 & \text{for all } j \in D, \\
& & x_{ij} \leqslant y_i & \text{for all } i\in F, j\in D, \\
& & x_{ij} \in \{0,1\} & \text{for all } i\in F, j\in D, \\
& & y_i \in \{0,1\} & \text{for all } i \in F.
\end{array}
$$

Here, F is the set of facilities and D is the set of clients.

(a) (2pt.) Explain in words what the inequality $x_{ij} \leqslant y_i$ means for the UFL problem.

(b) (5pt.) Suppose that after solving the LP-relaxation for some instance you find that for each client j there are at most 3 variables x_{ij} with $x_{ij} > 0$. In other words, in the support graph the degree of each point $j \in D$ is at most 3. Argue that it is easy to find a solution for the ILP with value at most $3Z$, where Z is the optimal value to the LP-relaxation.

Question 6 (8 points)
The figure shows 4 vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ originating from the center of a cube. For any pair of vectors v_i, v_j the angle between them has the same value α. It holds that $\cos(\alpha) = -1/3$ and $\alpha/\pi > 0.6$.



Let $G = (V,E)$ be a 4-colorable graph.

(a) (3pt.) Give the vector program that was used for 3-colorable graphs in Chapter 6. (For 3-colorable graphs, the optimal value of this VP is at most -0.5.) Argue that for 4-colorable graphs the optimal value is at most $-1/3$.

(b) (5pt.) Show how an optimal solution to the vector program can be used to find a random cut that contains more than 60 percent of the edges of G in expectation.